L^1-contraction for bounded (nonintegrable) solutions of degenerate parabolic equations

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We consider bounded nonintegrable entropy solutions of the following degenerate parabolic equation:

$$\begin{cases} \partial_t u + \operatorname{div} f(u) = L\varphi(u) & \text{in} \quad \mathbb{R}^d \times (0,T) \\ u(x,0) = u_0(x) & \text{on} \quad \mathbb{R}^d. \end{cases} \tag{DPE}$$

The operator L is either the x-Laplacian $\triangle$ or the nonlocal anomalous diffusion operator $\mathcal{L}^\mu$ defined on $C_c^\infty(\mathbb{R}^d)$ as

$$\mathcal{L}^\mu[\phi](x) = \int_{|z|>0} \phi(x+z) - \phi(x) - z \cdot D\phi(x)\mathbf{1}_{|z|\le 1} \,\mathrm{d}\mu(z).$$

We assume that $f \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^d)$, $\varphi \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$ is nondecreasing, and μ is a nonnegative Radon measure on $\mathbb{R}^d \setminus \{0\}$ such that

$$\int_{|z|\le 1} |z|^2 \,\mathrm{d}\mu(z) + \int_{|z|>1} \mathrm{d}\mu(z) < \infty.$$

Our main result is given below.

Theorem 1. *Let $B(x_0,R)$ denote the ball in $\mathbb{R}^d$ with center x_0 and radius R, and u, v be entropy solutions of* (DPE) *with respective initial data $u_0, v_0 \in L^\infty(\mathbb{R}^d)$. Then there exists $\psi \in C([0,T]; L^1(\mathbb{R}^d))$ such that for any $t \in [0,T]$*

$$\int_{B(x_0,R)} (u-v)^+(x,t)\,\mathrm{d}x \le \int_{\mathbb{R}^d} (u_0 - v_0)^+(x)\psi(x,t)\,\mathrm{d}x.$$

The function ψ is roughly speaking the solution of some second order Hamilton-Jacobi-Bellman equation which is a "dual equation" of (DPE). Our proof thus relies on a nonstandard L^1-estimate for that kind of equation. The above result is a sort of L^1_{loc}-contraction result, which implies the well-posedness in the L^∞ framework. Further consequences, are L^1_{loc}-and BV_{loc}-bounds, as well as, a comparison principle and L^∞-bounds. The presentation is mostly based on [3], see also [1, 2] for extensions and improvements.

Joint works with Nathaël Alibaud and Espen R. Jakobsen.

References

[1] N. Alibaud, J. Endal and E. R. Jakobsen. On some duality relations between Hamilton-Jacobi-Bellman and degenerate parabolic equations. Part I: The local case. Work in progress, 2016.

[2] N. Alibaud, J. Endal and E. R. Jakobsen. On some duality relations between Hamilton-Jacobi-Bellman and degenerate parabolic equations. Part II: The nonlocal case. Work in progress, 2016.

[3] J. Endal and E. R. Jakobsen. L^1 Contraction for Bounded (Non-integrable) Solutions of Degenerate Parabolic Equations. SIAM J. Math. Anal., 46(6):3957-3982, 2014.