Fractional Laplacian: explicit calculations and applications

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After a very short discussion of ten equivalent definitions of the fractional Laplace operator $L = (-\Delta)^{\alpha/2}$ (see [4]), I will present explicit expressions for Lf for a wide class of functions f, together with some applications. The main results are (see [1]):

- If G is the *Meijer G-function* with some parameters, then:
$$L[G(|x|^2)] = 2^\alpha \tilde{G}(|x|^2),$$
where $\tilde{G}$ is also the Meijer G-function, with appropriately modified parameters.
- If V is a *solid harmonic polynomial* of degree l in $\mathbf{R}^d$, then:
$$L[V(x)g(|x|^2)] = V(x)\tilde{g}(|x|^2)$$
for some $\tilde{g}$, and furthermore $L[g(|y|^2)] = \tilde{g}(|y|^2)$ in dimension $d + 2l$.

The latter result is a consequence of *Bochner's relation* for the Fourier transform, and it sometimes allows one to restrict attention to the class of radial functions. The former statement includes most previously known explicit expressions for Lf: many functions can be expressed as Meijer G-functions, including (generalised) hypergeometric, Bessel, Hermite and Airy functions or Jacobi polynomials.

The above two results can be used, for example, to develop highly efficient numerical methods for various variational problems involving the fractional Laplace operator. For example, if P is a member of the family of polynomials orthogonal with respect to the weight function $(1 - |x|^2)_+^{\alpha/2}$, then:
$$L[(1 - |x|^2)_+^{\alpha/2} P(x)] = c_{d,\alpha,n} P(x) \qquad \text{for } x \text{ in the unit ball,}$$
where n is the degree of P. (The family of orthogonal polynomials can be written explicitly in terms of Jacobi polynomials and solid harmonic polynomials.) This property has already been used to find very accurate numerical bounds, as well as some analytical estimates, for the eigenvalues of L in the unit ball (see [2]).

In the final part I will mention an explicit expression $\varphi(x) = \sin(x + \frac{(2-\alpha)\pi}{8}) - r(x)$, where r is an appropriate completely monotone function, for the solution of the eigenvalue problem in the half-line:
$$\begin{cases} L\varphi(x) = \varphi(x) & \text{for } x > 0, \\ \varphi(x) = 0 & \text{for } x \le 0. \end{cases}$$
This has found applications in fluctuation theory of symmetric stable processes. In fact our results cover also the non-symmetric case (see [3]).

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REFERENCES

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