Nonlinear and Nonlocal Degenerate Diffusions on Bounded Domains

Matteo Bonforte
Autonomous University of Madrid, Spain

We investigate quantitative properties of nonnegative solutions $u(t,x) \geq 0$ to the nonlinear fractional diffusion equation, $\partial_t u + \mathcal{L}F(u) = 0$ posed in a bounded domain, $x \in \Omega \subset \mathbb{R}^N$, with appropriate homogeneous Dirichlet boundary conditions. As $\mathcal{L}$ we can use a quite general class of linear operators that includes the three most common versions of the fractional Laplacian $(-\Delta)^s$, $0 < s < 1$, in a bounded domain with zero Dirichlet boundary conditions, but it also includes many other examples, since our theory only needs some basic properties that are typical of "linear heat semigroups". The nonlinearity F is assumed to be increasing and is allowed to be degenerate, the prototype being $F(u) = |u|^{m-1}u$, with $m > 1$. We will present some recent results about existence, uniqueness and a priori estimates for a quite large class of very weak solutions, that we call weak dual solutions. We then show sharper lower and upper estimates up to the boundary, which fairly combine into various forms of Harnack type inequalities. The standard Laplacian case $s = 1$ is included and the linear case $m = 1$ can be recovered in the limit in most of the results. When the nonlinearity is of the form $F(u) = |u|^{m-1}u$, with $m > 1$, global Harnack estimates are the key tool to understand the sharp asymptotic behaviour of the solutions. We finally show that solutions are classical, and even C^∞ in space in the interior of the domain, when the operator $\mathcal{L}$ is the (restricted) fractional Laplacian.
The above results are contained on a series of recent papers [1, 3, 5, 4, 2] in collaboration with J. L. Vázquez, and also with Y.Sire, A. Figalli, and X. Ros-Oton.

References

[1] M. Bonforte, J. L. Vázquez, *A Priori Estimates for Fractional Nonlinear Degenerate Diffusion Equations on bounded domains*, Arch. Rat. Mech. Anal. **218** (2015), no. 1, 317–362.

[2] M. Bonforte, Y. Sire, J. L. Vázquez,*Existence, Uniqueness and Asymptotic behaviour for fractional porous medium equations on bounded domains.* Discr. Cont. Dyn. Sys. **35** (2015), 5725–5767.

[3] M. Bonforte, J. L. Vázquez,*Fractional Nonlinear Degenerate Diffusion Equations on Bounded Domains Part I. Existence, Uniqueness and Upper Bounds.* Nonlin. Anal. TMA **131** (2016), 363–398.

[4] M. Bonforte, A. Figalli, X. Ros-Oton, *Infinite speed of propagation and regularity of solutions to the fractional porous medium equation in general domains*, To Appear in Comm. Pure Appl. Math (2016). http://arxiv.org/abs/1510.03758

[5] M. Bonforte, J. L. Vázquez,*Fractional Nonlinear Degenerate Diffusion Equations on Bounded Domains Part II. Positivity, Boundary behaviour and Harnack inequalities.* In preparation (2016).