Heat kernels of unimodal Lévy processes with weak scaling conditions

Michał Ryznar

Wrocław University of Science and Technology, Poland

Let $X(t)$ be an isotropic Lévy process on $\mathbb{R}^d$. Its transition density (*heat kernel*) $p_t(x)$ is assumed to be unimodal, that it is radial and decreasing function of the space variable, which is equivalent to the fact that its Lévy measure has radial and unimodal density. The most important example is the isotropic α-stable process, $0 < \alpha < 2$ with the fractional Laplacian $(\Delta)^{\alpha/2}$ as a generator. Another important class of such processes are subordinate Brownian motions with the generator $\psi(\Delta)$, where ψ is the Laplace exponent of the underlying subordinator.

In the talk we describe the behaviour of the transition density $p_t(x)$ under some weak type scaling condition expressed in terms of the Lévy–Khintchine exponent ψ of the process:

$$Ee^{i\langle\xi,X(t)\rangle} = e^{-t\psi(\xi)}, \quad \xi \in \mathbb{R}^d.$$

Here ψ is a radial function, which defines $\psi(\theta) = \psi(\xi)$ for $\xi \in \mathbb{R}^d$, $|\xi| = \theta$.

The scaling conditions are understood as follows. We say that ψ satisfies the weak *lower* scaling condition at infinity (WLSC) if there are numbers $\alpha > 0$, $\theta \geq 0$, such that

$$\inf_{y \geq x > \theta} \frac{\psi(y)}{\psi(x)} \left(\frac{x}{y}\right)^{\alpha} > 0.$$

We write $\psi \in$ WLSC.

The weak *upper* scaling condition at infinity (WUSC) means that there are numbers $\beta < 2$, $\theta \geq 0$, such that

$$\sup_{y \geq x > \theta} \frac{\psi(y)}{\psi(x)} \left(\frac{x}{y}\right)^{\beta} < \infty.$$

In short, $\psi \in$ WUSC.

The scaling conditions are global if we can take $\theta = 0$.

The main result regarding the description of the heat kernel will be the following estimate obtained in [1]:

$$p_t(x) \approx \ \psi^{-1}(1/t)^d \ \wedge \ \frac{t\,\psi(1/|x|)}{|x|^d}, \tag{1}$$

which holds locally (small t and x) or globally, contingent on the local or global weak scaling property of the symbol ψ.

Next, we study estimates the *Dirichlet heat kernel* p_D of smooth open sets $D \subset \mathbb{R}^d$ that is the transition density of the process killed on exiting D. The estimates have a form of explicit factorization involving the transition density p of the Lévy process (on the whole of $\mathbb{R}^d$), and the *survival probability* $\mathbb{P}^x(\tau_D > t)$, where τ_D is the first exit time of the process from D. For instance, if ψ has global lower and upper scalings, and D is a C^2 halfspace-like open set, then

$$p_D(t,x,y) \ \approx \ \mathbb{P}^x(\tau_D > t)\ p_t(x-y)\ \mathbb{P}^y(\tau_D > t), \qquad t > 0,\ x, y \in \mathbb{R}^d,$$

where $p_t(x-y)$ is estimated as above in (1) and

$$\mathbb{P}^x(\tau_D > t) \ \approx \ 1 \ \wedge \ \frac{1}{\sqrt{t\,\psi(1/\delta_D(x))}}.$$

Similar bounds, obtained in [2], will be presented for smooth bounded or exterior domains.

Joint work with Krzysztof Bogdan and Tomasz Grzywny.

References

[1] Krzysztof Bogdan, Tomasz Grzywny and Michał Ryznar, *Density and tails of unimodal convolution semigroups*, J. Funct. Anal. **266** (2014), no. 6, 3543–3571.

[2] Krzysztof Bogdan, Tomasz Grzywny and Michał Ryznar, *Dirichlet heat kernel for unimodal Lévy processes*, Stochastic Processes and their Applications **124** (2014), 3612—3650