Spatial asymptotics at infinity for heat kernels of some nonlocal operators

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We study a convolution semigroup of probability measures $\{P_t,\, t \geq 0\}$ on $\mathbb{R}^d$, $d \in \{1, 2, ...\}$, determined by their Fourier transforms $\mathcal{F}(P_t)(\xi) = \int_{\mathbb{R}^d} e^{i\xi\cdot y} P_t(dy) = \exp(-t\Phi(\xi))$, $t > 0$, with the Lévy-Khintchine exponent of the form

$$\Phi(\xi) = -i\xi \cdot b + \frac{1}{2}\xi \cdot A\xi + \int \left(1 - e^{i\xi\cdot y} + i\xi \cdot y 1_{B(0,1)}(y)\right) \nu(dy), \quad \xi \in \mathbb{R}^d,$$

where $b \in \mathbb{R}^d$ is a drift term, $A = (a_{ij})$ is symmetric, positive definite matrix and ν is a Lévy measure on $\mathbb{R}^d \setminus \{0\}$. The generator of the semigroup $\{P_t,\, t \geq 0\}$ is of the form

$$\begin{aligned} Lf(x) &= b \cdot \nabla f(x) + \frac{1}{2}\sum_{j,k=1}^{d} a_{jk} \frac{\partial^2 f}{\partial x_j \partial x_k}(x) \\ &\quad + \int_{\mathbb{R}^d\setminus\{0\}} (f(x+u) - f(x) - \nabla f(x) \cdot u 1_{B(0,1)}(u))\, \nu(du), \end{aligned}$$

for $f \in C_b^2(\mathbb{R}^d)$, and we have $\partial_t p_t(x) = Lp(t,x)$, $t > 0, x \in \mathbb{R}^d$, where $p_t(x)$ is the transition density of the semigroup $\{P_t,\, t \geq 0\}$.

The following theorem is a main result of this reseach.

Theorem 1. *Assume that* $A \equiv 0$ *or* $\inf_{|\xi|=1} \xi \cdot A\xi > 0$, $\nu(dx) = \nu(x)dx$ *and there exists a nonincreasing function* $f : (0, \infty) \to (0, \infty)$ *such that* $\nu(x) \asymp f(|x|)$, $x \in \mathbb{R}^d \setminus \{0\}$, *and*

$$G(r) := \sup_{|x|>1} \frac{\int_{\substack{|x-y|>r \\ |y|>r}} f(|x-y|)f(|y|)dy}{f(|x|)} \searrow 0 \qquad \text{as} \quad r \to \infty. \tag{1}$$

Moreover, suppose that $\nu(\mathbb{R}^d \setminus \{0\}) = \infty$ *and*

(i) *there exist a nonempty and bounded set* $T \subset (0, \infty)$ *and a constant* $C > 0$ *such that*

$$\int_{\mathbb{R}^d\setminus\{0\}} e^{-t\Re\Phi_\nu(\xi)} |\xi| d\xi \leq C[h(t)]^{-d-1}, \quad t \in T,$$

where $h(t) = 1/\Psi^{-1}(1/t)$, *for* $\Psi(r) = \sup_{|\xi|<r} Re(\Phi(\xi) - \xi \cdot A\xi)$, *and*

(ii) *there exist* $\theta \in \mathbb{S}^{d-1}$ *and* $\kappa \geq 0$ *such that*

$$\lim_{r\to\infty} \frac{\nu(r\theta - y)}{\nu(r\theta)} = e^{\kappa(\theta\cdot y)}, \quad y \in \mathbb{R}^d. \tag{2}$$

Then, for every $t \in T$ *and* $y \in \mathbb{R}^d$,

$$\lim_{r\to\infty} \frac{p_t(r\theta - y)}{t\, \nu(r\theta)} = e^{-t\widetilde{\Phi}(\kappa\theta) + \kappa(\theta\cdot y)}, \tag{3}$$

where

$$\widetilde{\Phi}(\xi) = -\xi \cdot b + \xi \cdot A\xi + \int_{\mathbb{R}^d\setminus\{0\}} \left(1 - e^{\xi\cdot y} + \xi \cdot y 1_{B(0,1)}(y)\right) \nu(y)\, dy, \quad \xi \in \mathbb{R}^d. \tag{4}$$

Moreover, if the convergence in (2) *of (ii) is compact, then* (3) *is uniform in* (t, y) *on each rectangle* $T \times B(0, \varrho)$, $\varrho > 0$.

Joint work with Kamil Kaleta.