Symmetries of solutions to nonlocal problems

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We investigate a general class of nonlocal problems in open sets Ω of the form

$$(P) \qquad \begin{cases} Iu = f(u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \\ \lim\limits_{|x|\to\infty} u(x) = 0 \end{cases}$$

where $f \in C^1(\mathbb{R})$ and I is a nonlocal operator with the principal value representation

$$I\varphi(x) = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^N\setminus B_\varepsilon(x)} (\varphi(x) - \varphi(y))J(|x-y|)\, dy \qquad \text{for } \varphi \in C_c^2(\mathbb{R}^N).$$

Here $J : (0,\infty) \to [0,\infty)$ is decreasing such that $J|_{(0,\delta)}$ is strictly decreasing for some $\delta > 0$,

$$\int_{\mathbb{R}^N} \min\{1, |z|^2\} J(|z|)\, dz < \infty \quad \text{and} \quad \int_{\mathbb{R}^N} J(|z|)\, dz = \infty.$$

Special cases are $I = (-\Delta)^s$, for $s \in (0,1)$ or $I = (\mathrm{id} - \Delta)^s - \mathrm{id}$, but also operators of zeroth order, e.g. with $J(r) = 1_{(0,1]}(r) r^{-N}$, are admissable. In the talk we will present special cases from [3, 4, 5, 6], which describe difference between symmetry results of the nonlocal problem (P) and of the classical problem with $I = -\Delta$ (see [1, 2]).
The first difference is that a priori one does not need to assume positivity:

Theorem 1. *Let $\Omega \subset \mathbb{R}^N$ be a ball and let u be a bounded nonnegative solution of (P). Then u is radially symmetric. Moreover, either $u \equiv 0$ on $\mathbb{R}^N$ or u is strictly decreasing in its radial direction and hence $u > 0$ in B.*

The second difference is that Ω may consist of more than one connected component:

Theorem 2. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be a radial set and let u be a bounded solution of (P). Assume there is a half space $H \subset \mathbb{R}^N$ with $0 \in \partial H$ and such that $u \geq u \circ Q_H$ in H, $u \not\equiv u \circ Q_H$, where Q_H is the reflection at ∂H. Then u is foliated Schwarz symmetric, i.e. there is $p \in S^{N-1}$ such that u is axially symmetric w.r.t. $\mathbb{R} \cdot p$ and nonincreasing in the polar angle $\theta = \arccos\left(\frac{x\cdot p}{|x|}\right)$.*

Theorem 1 is a joint work with the Tobias Weth.

References

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