Supersolutions in fractional nonlinear problems

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In this talk we consider the fractional p-Laplace type operator given by

$$\mathcal{L}u(x) = p.v. \int_{\mathbb{R}^n} |u(x) - u(y)|^{p-2}(u(x) - u(y))K(x,y)\,dy\,, \qquad x \in \mathbb{R}^n,$$

where $K(x,y) \approx |x-y|^{-n-sp}$, $s \in (0,1)$ and $p > 1$. We are interested in different notions of solutions, namely

(1) *Weak* solutions. These arise naturally as minimizers of the Gagliardo seminorm, i.e., minimizers of

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |u(x) - u(y)|^p K(x,y)\,dxdy,$$

and weak solutions are defined via first variation.

(2) Potential-theoretic (s,p)-*harmonic* functions. These are defined via comparison with weak solutions and they naturally arise for example in the Perron method.

(3) *Viscosity* solutions. The notion of viscosity solutions is based on the pointwise evaluation of the principal value for suitable approximations of solutions.

There are natural counterparts for corresponding sub- and supersolutions.

The purpose of the talk is to introduce main concepts of the theory and to give an overview of needed techniques. Indeed, we will discuss several results for the weak supersolutions, such as comparison principles, a priori bounds, lower semicontinuity, and many others. We will then describe a good definition of (s,p)-superharmonic functions, also discussing some related properties of them. These allow us to establish the nonlocal counterpart of the celebrated Perron method. In fact, with the correct definitions in hand, one our main results states that for bounded solutions, all the three different notions above coincide. There are surprisingly many technical tools needed. In particular, one needs to establish basic regularity theory for the related obstacle problem, and for these one needs for example suitable Harnack estimates for both solutions and supersolutions.

The talk is based on joint works with Janne Korvenpää, Erik Lindgren and Giampiero Palatucci