Strongly nonlinear multiplicative inequalities involving nonlocal operators

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Outlines

Introduction

We are interested in inequality:

$$\int_{(a,b)} |f^{'}(x)|^p h(f(x))dx \leq$$

$$\left(\sqrt{p-1}\right)^p \int_{(a,b)} \left(\sqrt{|f''(x)\mathcal{T}_h(f(x))|}\right)^p h(f(x))dx, \qquad (1)$$

and its generalizations.

Assumptions:

- $-\infty \leq a < b \leq +\infty$, $p \geq 2$,
- $f \in \mathcal{R}$, $C_0^\infty(a,b) \subseteq \mathcal{R} \subseteq W_{loc}^{2,1}(a,b)$, sometimes we assume $f \geq 0$,
- $h : \mathbf{R} \to \mathbf{R}_{\geq 0}$ is continuous (can be defined on $\mathbf{R}_{\geq 0}$ provided $f \geq 0$),
- $\mathcal{T}_h(\cdot)$ is continuous, interpreted as the transformation of f:

$$\mathcal{T}_h(\lambda) := \begin{cases} \frac{H(\lambda)}{h(\lambda)} & \text{if} \quad h(\lambda) \neq 0, \\ 0 & \text{if} \quad h(\lambda) = 0, \end{cases}$$

  where H is primitive to h,

Our motivations:

- Apply the new inequalities to singular PDEs.

Example models

- Thomas-Fermi model (1927, describes electric charge in isolated neutral atom)

$$\begin{cases} y''(t) = t^{\frac{1}{2}} y(t)^{\frac{3}{2}}, \ t \in (0,\infty), \\ y(0) = 0, \lim_{t\to\infty} y(t) = 0. \end{cases}$$

- Emden-Fowler problem (fluid dynamics):

$$\begin{cases} y'' + \lambda q(x) y^{-\gamma} = 0, \ x \in (0,1), \gamma > 0 \\ y(0) = y(1) = 0, \end{cases}$$

- model of membrana and model of mikro-electro-mechanical system (MEMS), papers by Esposito and coauthors

$$\begin{cases} -\Delta u = \frac{\lambda f(x)}{(1-u)^2} & \text{in} & \Omega \subseteq \mathbf{R}^2 \\ 0 \le u < 1 & \text{in} & \Omega \\ u = 0 & \text{on} & \partial\Omega \end{cases}$$

- models in cosmology, e.g. Makutuma model

$$\Delta u + \frac{1}{1+|x|^2} u^q = 0, \ x \in \mathbf{R}.$$

Unweighed simplest variant

Theorem (Katarzyna Pietruska-Paluba and A.K, 2006)

$$\int G(|\nabla u|)dx \quad \leq \quad C \int G(\sqrt{|u||\nabla^{(2)} u|})dx, \tag{2}$$

where G is convex.

Inequalities involving “weight”, d=1

Theorem (Jan Peszek and A.K., 2011)

$$\int_{(a,b)} |f^{'}(x)|^p h(f(x))dx \leq C \int_{(a,b)} \left(\sqrt{|f''(x)\mathcal{T}_h(f(x))|} \right)^p h(f(x))dx,$$

under certain assumptions on h and f.

The special case

Theorem (Jan Peszek and A.K., 2011)

Let $2 \leq p < \infty$, $\theta \in \mathbb{R}$ and $f \in W^{2,1}_{loc}(\mathbb{R})$ be such that $f^{'}$ has compact support. Assume additionally that at least one of the conditions is satisfied:

1. $\theta < \frac{1}{p}$,
2. $\theta > \frac{1}{p}$ *and f is nonnegative or (more generally) does not have isolated zeroes,*
3. $\theta > \frac{1}{p}$ *and there exists ϵ such that for every $r < R$:*

$$\int_{(r,R)\cap\{x:0<|f(x)|<\epsilon\}} \left(\frac{|f^{'}|}{|f|^{\theta}} \right)^p dx < \infty.$$

Then

$$\int_{\{x:f(x)\neq 0\}} \left(\frac{|f'|}{|f|^{\theta}}\right)^{p} dx \leq \left(\frac{p-1}{|1-\theta p|}\right)^{\frac{p}{2}} \int_{\{x:f(x)\neq 0\}} \left(\frac{\sqrt{|ff''|}}{|f|^{\theta}}\right)^{p} dx.$$

For $\theta = \frac{1}{2}$ and $f \geq 0$ we retrieve Mazja's inequality know earlier.

Generalization to Orlicz spaces

We consider certain set of assumptions:

(M) $M : [0, \infty) \to [0, \infty)$ is (convex) differentiable $N-$function, and M satisfies the condition:

$$d_M \frac{M(\lambda)}{\lambda} \leq M'(\lambda) \leq D_M \frac{M(\lambda)}{\lambda} \text{ for every } \lambda > 0, \tag{3}$$

where $D_M \geq d_M \geq 2$.

(h) $h : (0, \infty) \to (0, \infty)$ is locally Lipschitz and $H : (0, \infty) \to \mathbb{R}$ is its locally absolutely continuous primitive and $|h^{'} H|$ is well controlled by h^2 (precise formulation is omitted).

Theorem (Jan Peszek and A.K, 2012)

Assume that M satisfies **(M)**, $h : (0,\infty) \to (0,\infty)$ *satisfies* **(h)**. *Then any nonnegative $f \in W^{2,1}(\mathbb{R})$ such that f' has compact support satisfies inequality*

$$\int_{\mathbb{R}} M(|f'(x)|h(f(x)))dx \leq$$

$$C \int_{\mathbb{R}} M\left(\sqrt{|f''(x)\mathcal{T}_h(f(x))|} \cdot h(f(x))\right) dx.$$

This inequality has been applied to second order capacitary estimates and isoperimetric inequalities.

Applications to the nonlinear ODEs I

Consider the following O.D.E:

$$\begin{cases} f''(x) = g(x)\tau(f(x)) \text{ a.e. in } (a,b), \\ f \in \mathcal{R} \end{cases} \tag{4}$$

where $-\infty \le a < b \le +\infty$ and:

- $\tau : A \to \mathbb{R}$, $A \subseteq \mathbb{R}$ is an interval,
- $g \in L^q(a,b)$, $q \in [1,\infty]$,
- $f \in W^{2,1}_{loc}((a,b))$, $f(x) \in A$,
- set $\mathcal{R}$ defines the boundary conditions (admitted to our inequalities).

We find function $h(\cdot)$ such that

$$|g(x)|^q = \left|\frac{f^{''}(x)}{\tau(f(x))}\right|^q = |\mathcal{T}_h(f(x))f^{''}(x)|^{\frac{2q}{2}} h(f(x)),$$

We apply:

$$\int_{(a,b)} |f^{'}(x)|^{2q} h(f(x))dx \leq$$

$$\left(\sqrt{2q-1}\right)^{2q} \int_{(a,b)} \left(\sqrt{|f''(x)\mathcal{T}_h(f(x))|}\right)^{2q} h(f(x))dx.$$

We deduce that

$$\int |f^{'}|^{2q} h(f) \leq C\|g\|_q^q.$$

- Let $G = G_\tau$ be such transform of τ that $|(G(f))^{'}|^{2q} = |f^{'}|^{2q} \cdot h(f)$ $(G^{'} = h^{1/(2q)})$. Then $G(f) \in W^{1,2q}((a,b))$, so is λ-Hölder continuous, where $\lambda = 1 - \frac{1}{2q}$.
- we deduce the regularity and asymptotic behavior of solutions.

Application (with Jan Peszek, generalization with Katarzyna Mazowiecka)

Assumption: $1 \le q < \infty$, $\alpha \neq -1 + \frac{1}{q}$, $\kappa = -\mathrm{sign}(\alpha + 1 - \frac{1}{q})$, $0 < b \le \infty$, $g \in L^q(0, b)$ and let $f \in W^{2,1}_{loc}(0, b)$ and $u \ge 0$ solves:

$$f^{''}(x) = g(x)(f(x))^{\alpha} \text{ a.e. on } (0, b)$$

and nonlinear boundary condition (mixed type):

$$\liminf_{R \nearrow b} \kappa |f^{'}(R)|^{2q-2} f^{'}(R)(f(R))^{-q(\alpha+1)+1}$$
$$- \limsup_{r \searrow 0} \kappa |f^{'}(r)|^{2q-2} f^{'}(r)(f(r))^{-q(\alpha+1)+1} \le 0.$$

Theorem (Jan Peszek, A.K., 2011)

i)

$$\int_0^b |f'(x)|^{2q}|f(x)|^{-q(\alpha+1)}dx \le C_q \int_0^b |g(x)|^q dx,$$

ii)

$$\sup\left\{ \frac{|(f(x))^{\frac{1-\alpha}{2}} - (f(y))^{\frac{1-\alpha}{2}}|}{|x-y|^{1-\frac{1}{2q}}} : x, y \in (0,b) \right\} \le A_q \left(\int_0^b |g(x)|^q dx \right)^{\frac{1}{2q}},$$

iii) *If* $\alpha < 1$ *then* $\lim_{r\searrow 0} f(r) =: f(0)$ *exists and when* $f(0) = 0$

$$|f(x)|^{\frac{1-\alpha}{2}} \le A_q |x|^{1-\frac{1}{2q}} \left(\int_0^b |g(x)|^q dx \right)^{\frac{1}{2q}}.$$

Extensions were obtained with Katarzyna Mazowiecka.

Generalization to n-d (with Tomasz Choczewski)

We obtained the analogue of multiplicative inequality having the form:

$$\int_\Omega |\nabla f(x)|^p h(f(x))dx \leq$$
$$\left(\sqrt{p-1}\right)^p \int_\Omega \left(\sqrt{|\Delta f(x)\mathcal{T}_h(f(x))|}\right)^p h(f(x))dx, \tag{5}$$

and applications to the eigenvalue problems like:

$$\begin{cases} \Delta f(x) = g(x)\tau(f(x)) \text{ a.e. in } \Omega. \\ f \in \mathcal{R} \end{cases} \tag{6}$$

- we require information about second order vectorial Riesz transforms ($\frac{\partial}{\partial x_j} \circ \Delta^{-1/2}$, papers by Tadeusz Iwaniec and coauthors) and best constants in Hardy inequalities.
- inequality applies to the model of elektrostatic micromechanical systems (MEMS), which is reduced to the following problem

$$\begin{cases} \Delta u = \frac{\lambda f(x)}{(1-u)^2} & w \quad \Omega \\ u = 0 & on \quad \partial\Omega \\ 0 < u < 1 & in \quad \Omega \end{cases}$$

where $\lambda \geq 0$, $f \geq 0$, $u \in C^1(\overline{\Omega} \cap W^{2,2}(\Omega))$, and Ω is open and bounded (papers by Esposito).

Weighted variants in 1-d (with Ignacy Lipka)

$$\int_{(a,b)} |f^{'}(x)|^p h(f(x))\rho(x)dx \leq$$

$$C\left(\int_{(a,b)}\left(\sqrt{|f''(x)\mathcal{T}_h(f(x))|}\right)^p h(f(x))\rho(x)dx\right.$$

$$\left.+\int_{(a,b)}|\mathcal{T}_h(f(x))|^p h(f(x))|\rho^{'}(x)|dx\right)$$

Inequalities involving nonlocal operators (with Claudia Capogne and Alberto Fiorenza)

We obtain inequalities:

$$\int_{\mathbf{R}} M(|f^{'}(x)|h(f(x)))dx \leq$$

$$A \int_{\mathbf{R}} M\left(B \sqrt[p]{|\mathcal{M}f''(x)\mathcal{T}_{h,p}(f,x)|} \cdot h(f(x)) \right) dx,$$

$$\mathcal{T}_{h,p}(f,x) := \begin{cases} \frac{\int_{-\infty}^{x} \Phi_p\left(h(f(y))f^{'}(y)\right)\chi_{\{f(y)\neq 0\}}\, dy}{h(f(x))^{p-1}}, f(x) \neq 0 \\ 0, f(x) = 0 \end{cases}$$

and $\phi_p(s) = |s|^{p-2}s$. For $p = 2$ we have $\mathcal{T}_{h,p} = \mathcal{T}_h$, $\mathcal{M}h$ is Hardy - Littlewood maximal function.
The approach requires Simmonnenko and Boyed indeces.

Interpretation of the nonlinear transform

- For $h \equiv 1$ we have

$$\mathcal{T}_{h\equiv 1,p}(f,x) = \Delta^{-1}\Delta_p, \quad \Delta_p u = \mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right).$$

- In general

$$\mathcal{T}_{h,p}(f,x) = \frac{\Delta^{-1}\Delta_p(H(f))}{\Phi_p(h(f))}, \quad \Phi_p(s) = |s|^{p-2}s.$$

- In particular $\mathcal{T}_{h,p}(f,x)$ is nonlocal.

Example inequality

$$\int_{\mathbb{R}} |f^{'}(x)|^q (f(x))^{\alpha} dx \leq$$

$$C \int_{\mathbb{R}} \left(\mathcal{M}f^{''}(x)\right)^{\frac{q}{p}} \left(\int_{\infty}^{x} |f^{'}(y)|^{p-1} (f(y))^{\alpha(p-1)} dy\right)^{\frac{q}{p}} f(x)^{\frac{\alpha q}{p}}$$

$$\leq C \left(\int_{\mathbb{R}} \left(|f^{'}(y)|(f(y))^{\alpha}\right)^{p-1} dy\right)^{\frac{q}{p}} \int_{\mathbb{R}} \left(\mathcal{M}f^{''}(y)(f(y))^{\alpha}\right)^{\frac{q}{p}} dy.$$

The tools

Boyd indices and Simmonnenko indices: if the inequality holds with comvex function M then it holds with $M_1 \sim M$.

- First step: $M : [0, \infty) \to [0, \infty)$ is (convex) differentiable $N-$function and M satisfies the condition:

$$d_M \frac{M(\lambda)}{\lambda} \leq M'(\lambda) \leq D_M \frac{M(\lambda)}{\lambda} \text{ for every } \lambda > 0, \tag{7}$$

  where $D_M \geq d_M \geq 2$.
- Second step: we substitute M with $M_1 \sim M$ but smaller lower Simmonnenko index d_M. It leads to inequality with Boyd index $i_M := \inf_{M_1 \sim M} d_{M_1} < 2$.

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Thank you!