Boundary problems for nonlocal operators
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Nonlocal Operators and Partial Differential Equations, Będlewo 27.06-01.07.2016

Main notions
Let \( \Omega \) be a nonempty bounded open set in \( \mathbb{R}^n \) and let \( \nu : \mathbb{R}^n \to [0, \infty) \) be a symmetric Lévy measure, that is
\[
\nu(\{0\}) = 0, \quad \int_{\mathbb{R}^n} (1 + |x|^2)\nu(dx) < \infty, \quad \nu(A) = \nu(-A) \text{ for every Borel set } A.
\]
The main object of this presentation is the operator \( L_n(x) = \nu \int_{\mathbb{R}^n} \left( x(x) - \nu(x) \right) \delta_{x}(dx) \), for which we consider the Dirichlet problem of the form
\[
L_n u = f \text{ in } \Omega, \quad u = g \text{ outside } \Omega.
\]
In the sequel we assume that \( f \in L^2(\Omega) \).

Important facts
- If u \( \in C_0^0(\Omega) \), then \( \text{Lu exists for every } x \in \mathbb{R}^n \) and \( \text{Lu} \in L^2(\Omega) \).
- If \( u_1 = u_2 \) almost everywhere, then \( \text{Lu}_1 - \text{Lu}_2 = 0 \).

Function spaces - domains for weak solutions
For a nonempty open (not necessarily bounded) \( D \subseteq \mathbb{R}^n \), we define function spaces
\[
V_0^p(D) = \{ u \in L^p(D) : ||u||_{L^p(D)} < \infty \},
H_0^p(D) = \{ u \in V_0^p(D) : u = 0 \text{ a.e. in } \mathbb{R}^n \backslash D \},
\]
where
\[
||u||_{V_0^p(D)} = \left( \int_{\mathbb{R}^n} ||u(x)||_p^p dx \right)^{1/p},
||u||_{H_0^p(D)} = \left( \int_{\mathbb{R}^n} ||u(x)||_p^p dx \right)^{1/p}.
\]
Furthermore, we let \( H_0^p(\mathbb{R}^n) = V_0^p(\mathbb{R}^n) \). In particular, \( ||u||_{H_0^p(D)} = ||u||_{V_0^p(D)} \).

Important facts
- The norm in \( V_0^p(D) \) relaxes the “smoothness” assumption outside \( D \). This allows using less
  regular boundary conditions \( g \).
- The Dirichlet form (2) is well-defined in terms of \( L^p \) equivalence classes.
- If \( f = \int_{\mathbb{R}^n} \left( x(x) - \nu(x) \right) \delta_{x}(dx) \), then \( f \in H_0^p(\Omega) \).
- \( H_0^p(\mathbb{R}^n) \) is a Hilbert space, and \( H_0^p(D) \) is its closed subspace.
- \( \Theta \) Compactly supported Lipschitz functions are in \( H_0^p(\mathbb{R}^n) \).

Weak solutions
We say that u \( \in V_0^p(D) \) is a weak solution of (1), if \( u = g \) outside \( \Omega \), and for every \( \phi \in H_0^p(D) \)
\[
\int_{\mathbb{R}^n} \left( x(x) - \nu(x) \right) \delta_{x}(dx) \phi = \int_{\Omega} f \phi.
\]
Problem: we either need to assume that the exterior condition \( g \) is already given on the whole of \( \mathbb{R}^n \) or formulate some terms under which a function given in \( \Omega \) can be extended to a \( V_0^p(\mathbb{R}^n) \) function.

Important facts
- If \( u \in C_0^0(\Omega), \lambda \in \mathbb{R} \) and \( u + \lambda \) is a solution to (1), then it is also a weak solution.
- \( u \in V_0^p(\mathbb{R}^n) \) is a weak solution if and only if it minimizes the following energy functional
  among functions equal to \( g \) outside \( \Omega \)
\[
E(u) = \frac{1}{2} \int_{\mathbb{R}^n} \left( x(x) - \nu(x) \right) \delta_{x}(dx) \phi = \int_{\Omega} f \phi.
\]

Theorem on the existence and uniqueness of solutions
If \( g \) can be extended to a \( V_0^p(\mathbb{R}^n) \) function, and
- \( g \) is an atom, or
- \( g(B(0, \text{dist}(\partial\Omega)) \) > 0,
then the equation (1) has a unique weak solution.

Acknowledgements
This poster is based on my master thesis. Its structure, and the choice of contents were inspired
by the survey article by X. Ros-Oton [5].
I would like to express gratitude to my supervisor professor Krzysztof Bogdan for long and
fruitful discussions, as well as for motivating me to work.
I would also like to thank dr hab. Bartłomej Dyda, and dr Tomasz Grzywny for valuable
suggestions and remarks.

Poincaré inequality
The essence of proving the existence/uniqueness of solutions is to show that the Poincaré
inequality holds with \( C \) independent of \( u \). (c \Omega, \nu) \subseteq C[0, \|x\|_{L^p(\mathbb{R}^n)}].

Extension problem
The merit of the extension problem is to impose some conditions on \( D \subseteq \mathbb{R}^n \) and \( u : D \rightarrow \mathbb{R} \),
under which \( u \) can be extended to a \( V_0^p(\mathbb{R}^n) \) or \( \text{He}(\mathbb{R}^n) \) function.

Theorem
Let \( \Omega \) be a bounded \( C^{1,1} \) domain at scale \( \alpha \). Assume that \( d\nu(x) = \nu(x)dx \), and that there
exists \( C \) such that for every \( \beta \in [\alpha^{-1} \wedge \frac{1}{2}] \), we have \( \nu(x) \leq C\nu(x) \). If \( u : \Omega \rightarrow \mathbb{R} \)

Maximum principles
- \( \nu(\Omega - \Omega') > 0 \) if \( u \) is a weak solution with \( f, g \geq 0 \) then \( u \geq 0 \).

References