Boundary problems for nonlocal operators

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Main notions

Let Ω be a nonempty bounded open set in $\mathbb{R}^n$ and let $\nu : B(\mathbb{R}^n) \to [0,\infty]$ be a symmetric Lévy measure, that is

$$\nu(\{0\}) = 0, \quad \int_{\mathbb{R}^n} (1 \wedge |x|^2) d\nu(x) < \infty, \quad \nu(A) = \nu(-A) \text{ for every Borel set } A.$$

The main object of this presentation is the operator $Lu(x) = \mathrm{PV}\int_{\mathbb{R}^n} (u(x) - u(x+y)) d\nu(y)$, for which we consider the Dirichlet problem of the form

$$\begin{cases} Lu = f & \text{in } \Omega, \\ u = g & \text{outside } \Omega. \end{cases} \tag{1}$$

In the sequel we assume that $f \in L^2(\Omega)$.

Important facts

- If $u \in C_b^2(\mathbb{R}^n)$, then Lu exists for every $x \in \mathbb{R}^n$ and $Lu \in L^\infty(\Omega)$.
- If $u_1 = u_2$ almost everywhere, then $Lu_1 = Lu_2$ a.e.

Function spaces - domains for weak solutions

For a nonempty open (not necessarily bounded) $D \subseteq \mathbb{R}^n$, we define function spaces

$$V_\nu^D(\mathbb{R}^n) = \left\{ u \in L^0(\mathbb{R}^n) : ||u||_{V_\nu^D(\mathbb{R}^n)} < \infty \right\},$$
$$H_\nu^D(\mathbb{R}^n) = \left\{ u \in V_\nu^D(\mathbb{R}^n) : u \equiv 0 \text{ a.e. in } \mathbb{R}^n \backslash D \right\},$$

where

$$||u||_{V_\nu^D(\mathbb{R}^n)} = \sqrt{||u||^2_{L^2(D)} + \frac{1}{2} \int_D \int_{\mathbb{R}^n} (u(x) - u(y))^2 d\nu_y(x) dy}.$$

Furthermore, we let $H_\nu(\mathbb{R}^n) := V_\nu^{\mathbb{R}^n}(\mathbb{R}^n)$. In particular, $||u||_{H_\nu(\mathbb{R}^n)} = ||u||_{V_\nu^{\mathbb{R}^n}(\mathbb{R}^n)}$. For $u, v \in H_\nu(\mathbb{R}^n)$, we write

$$\langle u, v \rangle_\nu = \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (u(x) - u(y))(v(x) - v(y)) d\nu_y(x) dy \tag{2}$$

so that the norm on $H_\nu(\mathbb{R}^n)$ can be rewritten as $||u||_{H_\nu(\mathbb{R}^n)} = \sqrt{||u||^2_{L^2(\mathbb{R}^n)} + \langle u, u \rangle_\nu}$.

Important facts

- The norm in $V_\nu^D(\mathbb{R}^n)$ relaxes the "smoothness" assumption outside D. This allows using less regular exterior conditions g.
- The Dirichlet form (2) is well-defined in terms of L^2 equivalence classes.
- $\int_D \int_{\mathbb{R}^n} (u(x) - u(y))^2 d\nu_y(x) dy = \int_{\mathbb{R}^n} \int_D (u(x) - u(y))^2 d\nu_y(x) dy$.
- [2] $H_\nu(\mathbb{R}^n)$ is a Hilbert space, and $H_\nu^D(\mathbb{R}^n)$ is its closed subspace.
- [3] Compactly supported Lipschitz functions are in $H_\nu(\mathbb{R}^n)$.

Weak solutions

We say that $u \in V_\nu^\Omega(\mathbb{R}^n)$ is a weak solution of (1), if $u = g$ outside Ω, and for every $\phi \in H_\nu^\Omega(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (u(x) - u(y))(\phi(x) - \phi(y)) d\nu_x(y) dx = \int_\Omega f\phi.$$

Problem: we either need to assume that the exterior condition g is already given on the whole of $\mathbb{R}^n$, or formulate some terms under which a function given in Ω^c can be extended to a $V_\nu^\Omega(\mathbb{R}^n)$ ($H_\nu(\mathbb{R}^n)$) function.

Important facts

- If $u \in C_c^2(\mathbb{R}^n)$, $\lambda \in \mathbb{R}$, and $u + \lambda$ is a solution to (1), then it is also a weak solution.
- $u \in V_\nu^\Omega(\mathbb{R}^n)$ is a weak solution if and only if it minimizes the following energy functional among functions equal to g outside Ω

$$E(u) = \frac{1}{4} \iint_{\mathbb{R}^{2n} \backslash \Omega^C \times \Omega^C} (u(x) - u(y))^2 d\nu_y(x) dy - \int_\Omega fu.$$

Theorem on the existence and uniqueness of solutions

If g can be extended to a $V_\nu^\Omega(\mathbb{R}^n)$ function, and

- ν has an atom, or
- $\nu(B(0, \mathrm{diam}(\Omega))^c) > 0$,

then the equation (1) has a unique weak solution.
The case of existence and uniqueness of the solutions for absolutely continuous measures ν was resolved (positively) in the work of Felsinger, Kassmann and Voigt. [2].

Poincaré inequality

The essence of proving the existence/uniqueness of solutions is to show that the Poincaré inequality holds with C independent of u: $\langle u, u \rangle_\nu \leqslant C||u||_{L^2(\Omega)}$. Given this inequality, we can also show that weak solutions are stable under taking subspaces.

Extension problem

The merit of the extension problem is to impose some conditions on $D \subseteq \mathbb{R}^n$ and $u : D \longmapsto \mathbb{R}$, under which u can be extended to a $V_\nu^\Omega(\mathbb{R}^n)$ or $H_\nu(\mathbb{R}^n)$ function.

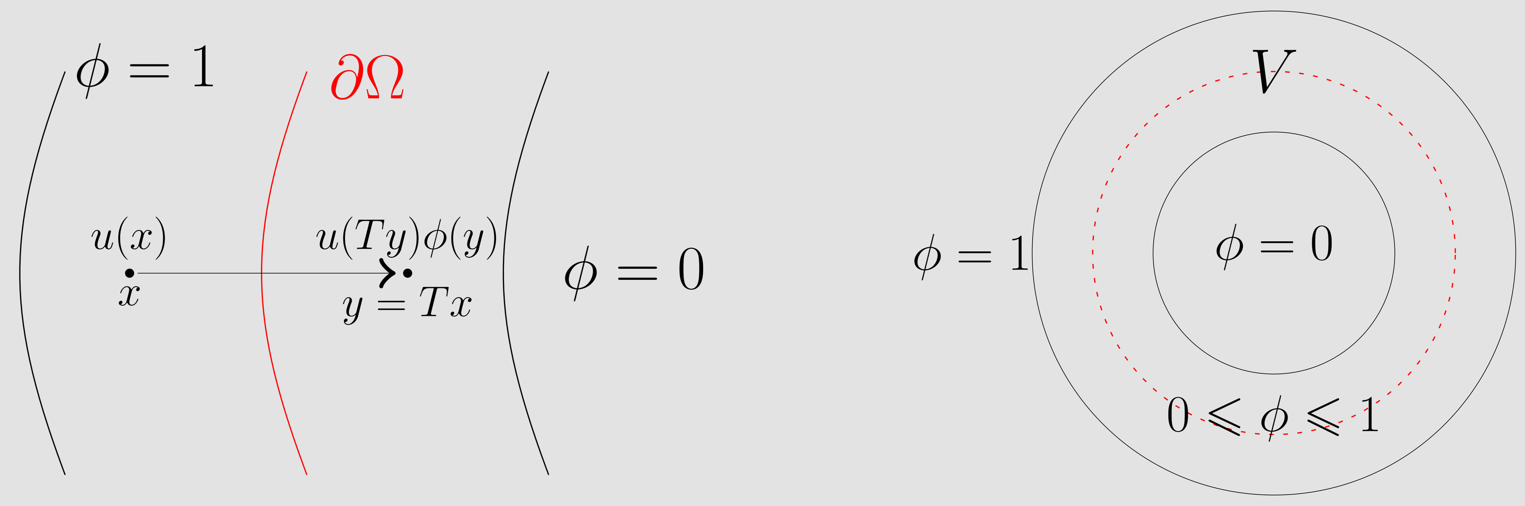
Theorem

Let Ω be a bounded $C^{1,1}$ domain at scale α. Assume that $d\nu(x) = v(|x|)dx$, and that there exists C_α such that for every $\beta \in [\alpha^{-1} \wedge \frac{1}{3}, \alpha]$, we have $v(\alpha x) \leqslant C_\alpha v(x)$. If $u : \Omega^c \longmapsto \mathbb{R}$ satisfies

$$\int_{\Omega^c} \int_{\Omega^c} (u(x) - u(y))^2 d\nu_y(x) dx < \infty, \tag{3}$$

then it can be extended to a $H_\nu(\mathbb{R}^n)$ function.

About the proof



- T - reflection through the boundary $\partial\Omega$ given on V - a neighborhood of Ω.
- $\phi \in C^\infty(\mathbb{R}^n)$: $0 \leqslant \phi \leqslant 1$, $\phi = 1$ in Ω^c, $\phi = 0$ in $\Omega \backslash V$.
- Purely geometric proof that T is Lipschitz.
- For $x \in V \cap \Omega$ put $u(x) = u(Tx)\phi(x)$, for $x \in \Omega \backslash V$, $u(x) = 0$.
- Integration by substitution theorem - Jacobian can be omitted when substituting $y = Tx$.
- Proof that (3) implies the finiteness of $H_\nu(\mathbb{R}^n)$ norm.

The extension problem for fractional Sobolev spaces was investigated a long time ago in [4]. Recently, Zhou [6] characterized the extension domains in that case.

Maximum principles

- (arbitrary ν) If u is continuous, $Lu \geqslant 0$ in Ω, and $u \geqslant 0$ outside Ω, then $u \geqslant 0$ in Ω.
- ($\nu((\Omega - \Omega)^c) > 0$) If u is a weak solution with $f, g \geqslant 0$, then $u \geqslant 0$.

About the proof

The proof of the strong maximum principle conceptually resembles the proof of its local version (for the Laplacian). It uses the fact that we can *jump out* from Ω in a finite number of steps. By using similar *jump out* arguments it may be possible to drop the assumptions on ν in the weak version.

Barriers

In [5], Ros-Oton presents the functions $w \in C_c^\infty(\mathbb{R}^n)$ satisfying $Lw \geqslant 1$ in Ω, $w \geqslant 0$, and $w \leqslant C$ in Ω. These so-called barriers are then used to obtain the following L^∞ bounds for the solution u

$$||u||_{L^\infty(\mathbb{R}^n)} \leqslant C||f||_{L^\infty(\mathbb{R}^n)} + ||g||_{L^\infty(\mathbb{R}^n)}.$$

We slightly refine the construction of barriers in order to see that if Ω_ε is the ε neighborhood of Ω, then

$$C^{-1} \geqslant \lim_{\varepsilon \to 0} \inf_{x \in \Omega} \kappa^{\Omega_\varepsilon}(x) := \lim_{\varepsilon \to 0} \inf_{x \in \Omega} \nu((\Omega_\varepsilon - x)^c).$$

A similar estimate - $\inf_{x \in \Omega} \kappa^\Omega(x)$ appears in [1] for certain perturbations of the fractional Laplacian. By considering a discrete Lévy measure $\sum_{k \in \mathbb{Z} \backslash \{0\}} \frac{1}{k^2} \delta_k$ with $\Omega = (0, 1)$ we find out that the expressions in both estimates are not always equal.

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