Eigenvalues of Fourier multipliers and generalized tight frames

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Dirichlet eigenvalues

Let λ_i be all solutions of

$$\Phi(-\Delta)u = \lambda u, \qquad \text{on } \Omega$$

with "zero boundary condition".

Then
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Somewhat more formally

Energy functional in Fourier domain:

$$\int_{\mathbb{R}^d} \Phi\left(|\xi|^2\right) |\widehat{u}|^2 d\xi,$$

for $u \in H_0^s(\Omega)$ (closure of C_c^∞), with s determined by Φ.
Eigenvalues are minimizers of the Rayleigh quotient:

$$R[u] = \frac{\int_{\mathbb{R}^d} \Phi\left(|\xi|^2\right) |\widehat{u}|^2 d\xi}{\int_{\mathbb{R}^d} |u(x)|^2 dx}$$

Examples:

Laplacian - eigenvalues of membranes

$\Phi(t) = t$, with $H_0^1(\Omega)$.
The boundary condition is enforced on $\partial\Omega$.

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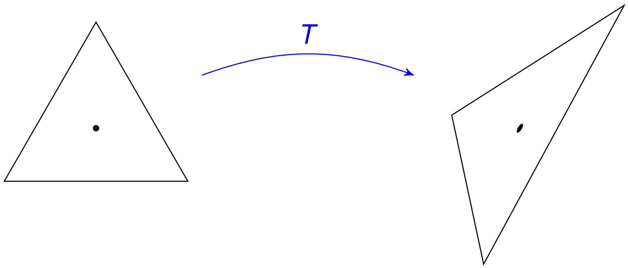
The boundary condition is enforced on Ω^c.

bi-Laplacian - plate eigenvalues

$\Phi(t) = t^2$, with $H^2_0(\Omega)$.
Two boundary conditions enforced on $\partial\Omega$ (function and its gradient vanish).

How do eigenvalues change under stretching of the domain?



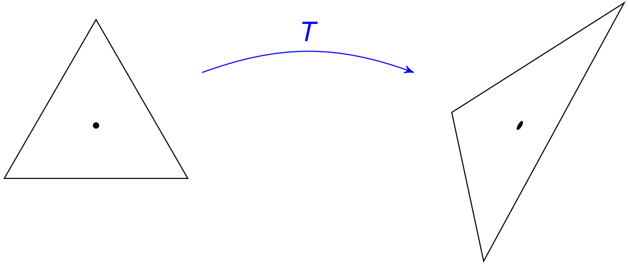
How do eigenvalues change under stretching of the domain?

Theorem (Siudeja, ACHA 2015)

Suppose Ω has N-fold rotational symmetry of order $N \geq 3$, T is a linear transformation, and Φ is concave. With $c_T = \|T^{-1}\|_{HS}^2/2$

$$\lambda_1 + \cdots + \lambda_n \quad \text{for } \Phi(|\xi|^2/c_T) \quad \text{on } T(\Omega)$$

is maximal when T is a multiple of identity.
($\|M\|_{HS}^2 = \sum M_{ij}^2$ - Hilbert-Schmidt norm)



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Corollary: Fractional Laplacian case

Let A and I be area and moment of inertia of a domain. For $\alpha \leq 2$

$$(\lambda_1 + \cdots + \lambda_k)^{2/\alpha} A \cdot \frac{A^2}{I}$$

is maximal on Ω, among all $T(\Omega)$. E.g. equilateral among triangles, square among parallelograms, disk among ellipses.

What are we getting?

- Sharp results with known extremizers.
- Easy to evaluate geometric factor (norm of the transformation matrix), which can be interpreted as natural quantities (area, moment of inertia).
- Transformed domains can be compared to the extremal domain even when the exact eigenvalues are not known for the extremizer. E.g. bound for eigenvalues of ellipses for the fractional Laplacian using numerical results by Dyda, Kuznetsov and Kwaśnicki.

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What would we like next?

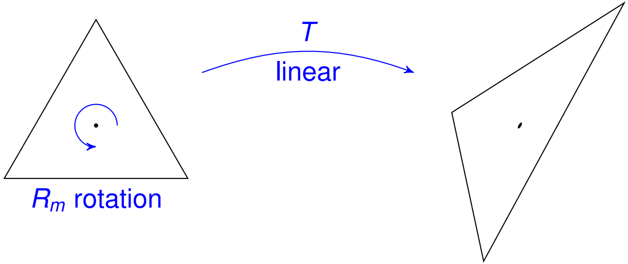
General results for fractional Laplacian on starlike domains, as in:

- Dirichlet/Neumann Laplacian: Laugesen-Siudeja, JFA 2011 (area-preserving instead of linear transformations)
- Steklov Laplacian: Laugesen-Girouard-Siudeja, ARMA 2016 (quasiconformal transformations)

Method of Rotations and Tight Frames (Laugesen-Siudeja 2011)

u: the first (unknown) eigenfunction on Ω for $\Phi(c|\xi|^2)$, $c = \|T^{-1}\|_{HS}^2/2$,
R_m: rotation by $2\pi m/N$,
$u \circ R_m \circ T^{-1}$: trial function for $\Phi(|\xi|^2)$ on $T(\Omega)$,

$$\lambda_1(T(\Omega)) \leq \int \Phi\left(|T^{-\dagger} R_m^\dagger \xi|^2\right) |\widehat{u}|^2 \, d\xi$$
$$= \int \Phi\left(\sum_k |T_k^{-\dagger} R_m^\dagger \xi|^2\right) |\widehat{u}|^2 \, d\xi$$



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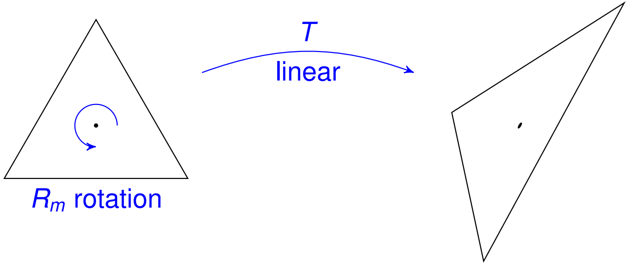

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&\leq \int \Phi\left(\sum_k \frac{1}{N} \sum_m |T_k^{-\dagger} R_m^\dagger \xi|^2\right) |\widehat{u}|^2 \, d\xi
\end{aligned}$$

We average over $m = 1, \ldots, N$ using Plancherel identity for rotational orbit $\{R_1\xi, \ldots, R_N\xi\}$ (Tight Frame) with $\eta = T_k^{-\dagger}$

$$\frac{1}{N} \sum_{m=1}^{N} |\eta \cdot (R_m^\dagger \xi)|^2 = \frac{1}{2} |\eta|^2 |\xi|^2.$$

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&= \int \Phi\left(\sum_k \frac{1}{2} |T_k^{-\dagger}|^2 |\xi|^2\right) |\widehat{u}|^2 \, d\xi \\
&= \int \Phi\left(c|\xi|^2\right) |\widehat{u}|^2 \, d\xi = \lambda_1(\Omega).
\end{aligned}$$

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Theorem (Siudeja, ARMA 2016)

Assume that Ω has N-fold rotational symmetry with $N \geq 3$ and $N \neq 4$. Then for bi-Laplacian ($\Phi(t) = t^2$)

$$(\lambda_1 + \cdots + \lambda_k)A^2 \cdot \frac{A^3}{I_4}$$

is maximized by Ω among all linearly transformed domains $T(\Omega)$. ($I_4 = \int_\Omega |x|^4 \, dx$ assuming center of mass at 0, 4th moment of mass; moment of inertia = 2nd moment)

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Note that the result is not true for parallelograms. Numerically, square is not a maximizer!

Method of Rotations and Tight Frames (bi-Laplacian case)

$$\lambda_1(T(\Omega)) \le \sum_i \int_{R^2} |T^{-\dagger} U_m^\dagger \xi|^4 |\widehat{u}|^2 \, d\xi = \sum_k \int_{R^2} |T_k^{-\dagger} U_m^\dagger \xi|^4 |\widehat{u}|^2 \, d\xi$$

Cannot split T into rows without introducing cross-terms!

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We average over $m = 1, \ldots, N$ using strange Plancherel identity for rotational orbit $\{U_1\xi, \ldots, U_N\xi\}$ (Tight 2-Frame) with $\eta = T_k^{-\dagger}$

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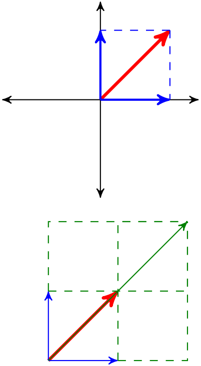
$$\lambda_1(T(\Omega)) \le c_T \sum_k \int_{R^2} |T_k^{-\dagger}|^4 |\xi|^4 |\widehat{u}_i|^2 \, d\xi$$

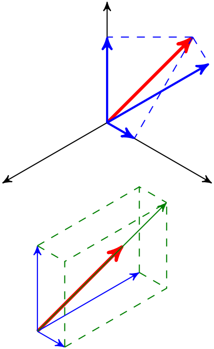
$$= C_T \sum_i \int_{R^2} |\xi|^4 |\widehat{u}_i|^2 \, d\xi = C_T \lambda_1(\Omega).$$

Pythagorean Theorem: N=4.



$$\sum_{m=1}^{4} |\eta \cdot (U_m \xi)|^2 = \frac{4}{2} |\eta|^2 |\xi|^2$$

Mercedes Tight Frame: N=3.



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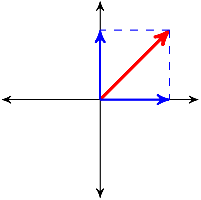

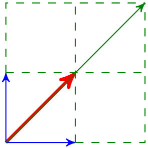

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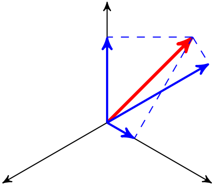

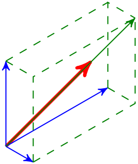

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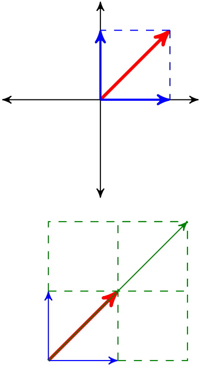

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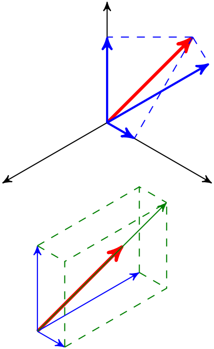

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$$\sum_{m=1}^{3} |\eta \cdot (U_m \xi)|^4 = \frac{3}{8} |\eta|^4 |\xi|^4$$

Groups of isometries that work with power p (even):

2D: N-th roots of unity:

$N = 3$ works with powers $p = 2, 4$.
$N = 5$ works with powers $p = 2, 4, 6, 8$.
$N = 6$ works with powers $p = 2, 4$.
$N = 7$ works with powers $p = 2, 4, 6, 8, 10, 12$.

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$N = 6$ works with powers $p = 2, 4$.
$N = 7$ works with powers $p = 2, 4, 6, 8, 10, 12$.
In general: if $p < N$ or ($p < 2N$ and N odd)

3D

Only orbits of the icosahedral group work with $p = 4$. Other regular solids only with $p = 2$.

4D

Group generated by 24-cell works with $p = 2, 4$.
Group generated 120-cell works with $p = 2, 4, 6, 8$.

What is going on here?

Definition

A polynomial P is invariant for a group of isometries G on R^d if for any $U \in G$ and $x \in R^d$ we have $P(Ux) = P(x)$.

$P(x) = |x|^p$ with even p is always invariant.

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Uniqueness implies Plancherel

Define

$$P(\xi) := \frac{1}{N} \sum_{U \in G} |\eta \cdot U\xi|^p$$

Due to the group property we easily get $P(V\xi) = P(\xi)$ and P is homogeneous. Uniqueness implies $P(\xi) = c_\eta |\xi|^p$.

Nonunique polynomial implies no Plancherel

Suppose there are two invariant quadratic polynomials for G, then there is an invariant linear function.

Take η from the hyperplane described by this linear equation, and ξ orthogonal the that hyperplane.

Then for any $U \in G$

$$\eta \cdot U\xi = 0.$$

Hence Plancherel's identity does not hold (left side is 0).

Generalized Plancherel's identity

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If

$$\sum_{U \in G} |\eta \cdot U\xi|^p = C_p |\eta|^p |\xi|^p.$$

is true for any η *and* ξ, *then*

$$\sum_{U \in G} |TU\xi|^p = C_{T,p} |\xi|^p.$$

holds for any matrix T, *and* $C_{T,p}$ *depends only on the singular values of* T *(streching factors).*

Constant $C_{T,p}$ can be computed explicitely.

$$\sum_{U \in G} |TU\xi|^p = C_{T,p}|\xi|^p.$$

Take the full isometry group of $\mathbb{R}^d$ (U is any unitary matrix) as G. Use singular value decomposition of T to discover that C_T only depends on the singular values of T, and not on G.

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Many pages later

C_T can be written as a nice linear combination of the Schatten norms of T, so the singular values are not needed.
(Schatten norms: $\|T\|_{2p}^{2p} = Tr((T^*T)^p)$, $p = 1$ gives Hilbert-Schmidt)