Semiconcavity of viscosity solutions for a class of degenerate elliptic integro-differential equations in $\mathbb{R}^n$

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June 30, 2016

Background

We fix a Polish space $\mathcal{A}$ (the control space) and a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, \mathbb{P})$. For any $(t,x) \in [0,+\infty) \times \mathbb{R}^n$ and any predictable process $\alpha(\cdot) : [0,+\infty) \times \Omega \to \mathcal{A}$, we consider

$$\begin{aligned} X(s) &= x + \int_t^s b(X(r),\alpha(r))dr + \int_t^s \sigma(X(r),\alpha(r))dW(r) \\ &\quad + \int_t^s \int_{\|z\|\leq 1} j(X(r),\alpha(r),z)\tilde{N}(dr,dz) \\ &\quad + \int_t^s \int_{\|z\|\geq 1} j(X(r),\alpha(r),z)N(dr,dz), \end{aligned}$$

where W is a standard m-dimensional Brownian motion and N is an independent Poisson random measure on $\mathbb{R}^+ \times (\mathbb{R}^n \setminus \{0\})$ with compensated measure $\tilde{N}$ and intensity measure μ.

Background

The value function of the infinite horizon problem is given by

$$V(x) = \inf_{\alpha(\cdot)} \mathbb{E}\Big[\int_0^{+\infty} e^{-\int_t^s c(X(\tau),\alpha(\tau))d\tau} f(X(s), \alpha(s))ds \Big]$$

and the corresponding nonlocal Bellman equation is of form

$$\sup_{\alpha \in \mathcal{A}} \{ -\frac{1}{2} Tr(\sigma_\alpha \sigma_\alpha^T D^2 u) - I_\alpha[\cdot, u] - b_\alpha \cdot Du + c_\alpha u - f_\alpha \} = 0, \quad (1)$$

where I_α is of the Lévy-Itô form, i.e.,

$$I_\alpha[x, u] = \int_{\mathbb{R}^n} [u(x+j_\alpha(x,z)) - u(x) - \mathbb{1}_{B_1(0)}(z) Du(x) \cdot j_\alpha(x,z)] \mu(dz).$$

Nonlocal Fully Nonlinear Equations

We consider

$$G(x, u, Du, D^2u, I[x, u]) = 0, \quad \text{in } \mathbb{R}^n. \tag{2}$$

The nonlinearity $G : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n \times \mathbb{R} \to \mathbb{R}$ is a continuous function which is coercive, i.e. there is a positive constant γ such that, for any $x, p \in \mathbb{R}^n$, $r \geq s$, $X \in \mathbb{S}^n$, $l \in \mathbb{R}$

$$\gamma(r - s) \leq G(x, r, p, X, l) - G(x, s, p, X, l),$$

and degenerate elliptic in a sense that, for any $x, p \in \mathbb{R}^n$, $r, l_1, l_2 \in \mathbb{R}$, $X, Y \in \mathbb{S}^n$

$$G(x, r, p, X, l_1) \leq G(x, r, p, Y, l_2) \quad \text{if } X \geq Y,\ l_1 \geq l_2.$$

I is of the Lévy-Itô form, i.e.,

$$I[x, u] := \int_{\mathbb{R}^n} [u(x + j(x, z)) - u(x) - \mathbb{1}_{B_1(0)}(z) Du(x) \cdot j(x, z)] \mu(dz).$$

Regularity results under uniform ellipticity assumption

- C^{α} regularity: Barles, Caffarelli, Chang, Imbert, Kassmann, Rang, Schwab, Silvestre, Song ...
- $C^{1+\alpha}$ regularity: Caffarelli, Chang, Kriventsov, Serra, Silvestre...
- $C^{\sigma+\alpha}$ regularity: Caffarelli, Chang, Dong, Jin, Serra, Silvestre, Xiong,...

Regularity results under degenerate ellipticity assumption

Few regularity results are known under degenerate ellipticity assumption.

- C^α regularity: Jakobsen, Karlsen.
- Semiconcavity: Jing, Feleqi.

Assumptions

To prove semiconcavity, we first need Hölder and Lipschitz regularity of viscosity solutions of (1) and (2). We make the following assumptions on the nonlinearity G and the function $j(x,z)$.

$(H1)$ For any $x, y \in \mathbb{R}^n$, we have

$$|j(x,z) - j(y,z)| \leq |x-y|\rho(z) \quad \text{for } z \in \mathbb{R}^n,$$

$$|j(0,z)| \leq \rho(z) \quad \text{for } z \in \mathbb{R}^n.$$

where $\rho : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^+$ is a Borel measurable function satisfying

$$\int_{\mathbb{R}^n \setminus \{0\}} \rho(z)^2 \mu(dz) < +\infty.$$

Assumptions

$(H2)$ There are a constant $0 < \theta \leq 1$, a non-negative constant Λ and a positive constant C such that, for any $x, y \in \mathbb{R}^n$, $r, l_x, l_y \in \mathbb{R}$, $X, Y \in \mathbb{S}^n$ and $L, \eta > 0$, we have

$$\begin{aligned} & G(y, r, L\theta|x-y|^{\theta-2}(x-y), Y, l_y) \\ & -G(x, r, L\theta|x-y|^{\theta-2}(x-y) + 2\eta x, X, l_x) \\ \leq \; & \Lambda(l_x - l_y) + C(1+L)|x-y|^{\theta} + C\eta(1+|x|^2), \end{aligned}$$

if

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq L|x-y|^{\theta-2} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 2\eta \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

An Example

We consider the nonlocal fully nonlinear equation

$$-Tr(\sigma\sigma^T D^2 u) + F(I[\cdot, u]) + b \cdot Du + cu + f = 0, \quad \text{in } \mathbb{R}^n,$$

where $F : \mathbb{R} \to \mathbb{R}$ is a continuous function. Suppose the following conditions are satisfied: there exists a non-negative constant Λ and γ such that, for any $l_x, l_y \in \mathbb{R}$,

$$c \geq \gamma \text{ in } \mathbb{R}^n,$$

$$\max\{[c]_{0,\theta;\mathbb{R}^n}, [\sigma]_{0,1;\mathbb{R}^n}, [b]_{0,1;\mathbb{R}^n}, [f]_{0,\theta,\mathbb{R}^n}\} < +\infty,$$

$$F(l_y) - F(l_x) \leq \Lambda(l_x - l_y).$$

Hölder and Lipschitz regularity

Theorem (C. Mou, Indiana Univ. Math. J. to appear)

Suppose that the nonlinearity G in (2) satisfies (H2), and that $j(x,z)$ satisfies assumption (H1). Then, if $u \in BUC(\mathbb{R}^n)$ is a viscosity solution of (2) and γ is sufficiently large, we have $u \in C^{0,\theta}(\mathbb{R}^n)$.

Hölder and Lipschitz regularity

Theorem (C. Mou, Indiana Univ. Math. J. to appear)

Suppose that $c_\alpha \geq \gamma$ *in* $\mathbb{R}^n$ *and that the family* $\{j_\alpha(x,z)\}$ *satisfies assumption (H1). Suppose moreover that for any* $0 < \theta \leq 1$

$$\sup_{\alpha \in \mathcal{A}} \max\{|\sigma_\alpha(0)|, |b_\alpha(0)|\} < +\infty,$$

and

$$\sup_{\alpha \in \mathcal{A}} \max\{[\sigma_\alpha]_{0,1;\mathbb{R}^n}, [b_\alpha]_{0,1;\mathbb{R}^n}, [c_\alpha]_{0,\theta,\mathbb{R}^n}, [f_\alpha]_{0,\theta,\mathbb{R}^n}\} < +\infty.$$

Then, if $u \in BUC(\mathbb{R}^n)$ *is a viscosity solution of (1) and* γ *is sufficiently large, we have* $u \in C^{0,\theta}(\mathbb{R}^n)$*.*

Assumptions

With Lipschitz regularity, we now have enough ingredients to prove semiconcavity of viscosity solutions of (1) and (2). We will impose the following conditions on G and $j(x,z)$.

$(\bar{H}1)$ $(H1)$ holds and, there is a constant $1 < \bar{\theta} \leq 2$ such that for any $x, y \in \mathbb{R}^n$ we have

$$|j(x,z) + j(y,z) - 2j(\frac{x+y}{2}, z)| \leq |x-y|^{\bar{\theta}} \rho(z) \quad \text{for } z \in \mathbb{R}^n.$$

$(\bar{H}2)$ If $\varphi \in C^{0,1}(\bar{\mathbb{R}}^n)$, there are a non-negative constant Λ and a positive constants C such that, for any $x, y, z \in \mathbb{R}^n$, $l_x, l_y, l_z \in \mathbb{R}$, $X, Y, Z \in \mathbb{S}^n$ and $L, \eta > 0$, we have

$$\begin{aligned} & 2G(z, \varphi(z), -\frac{L}{2} D_z \phi(x,y,z), \frac{Z}{2}, l_z) \\ & -G(x, \varphi(x), L D_x \phi(x,y,z) + 2\eta x, X, l_x) \\ & -G(y, \varphi(y), L D_y \phi(x,y,z), Y, l_y) \\ \leq \; & -\gamma\big(\varphi(x) + \varphi(y) - 2\varphi(z)\big) + \Lambda(l_x + l_y - 2l_z) \\ & +C(1+L)\phi(x,y,z) + C\eta(1+|x|^2), \end{aligned}$$

Assumptions

if

$$
\begin{pmatrix} X & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & -Z \end{pmatrix}
\leq \frac{L}{\phi(x,y,z)} \Big[\bar{\theta}(2\bar{\theta}-1)|x-y|^{2\bar{\theta}-2} \begin{pmatrix} I & -I & 0 \\ -I & I & 0 \\ 0 & 0 & 0 \end{pmatrix}
+ \begin{pmatrix} I & I & -2I \\ I & I & -2I \\ -2I & -2I & 4I \end{pmatrix} \Big] + 2\eta \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
$$

where $\bar{\theta}$ is given in $(\bar{H}1)$ and

$$
\phi(x,y,z) = (|x-y|^{2\bar{\theta}} + |x+y-2z|^2)^{\frac{1}{2}}.
$$

An Example

We consider the nonlinear convex nonlocal equation

$$-Tr(\sigma\sigma^T D^2 u) + F(I[\cdot, u]) + b \cdot Du + cu + f = 0, \quad \text{in } \mathbb{R}^n,$$

where $F : \mathbb{R} \to \mathbb{R}$ is a convex function. Suppose the following conditions are satisfied: there exists a non-negative constant Λ and γ such that, for any $l_x, l_y \in \mathbb{R}$,

$$c \geq \gamma \text{ in } \mathbb{R}^n \text{ and } c \in C^{1,\bar{\theta}-1}(\bar{\mathbb{R}}^n),$$

$$f \text{ is } \bar{\theta}\text{-semiconvex in } \mathbb{R}^n$$

$$\max\{[\sigma]_{0,1;\mathbb{R}^n}, [\sigma]_{1,\bar{\theta}-1;\mathbb{R}^n}, [b]_{0,1;\mathbb{R}^n}, [b]_{1,\bar{\theta}-1,\mathbb{R}^n}, [f]_{0,1,\mathbb{R}^n}\} < +\infty,$$

$$F(l_y) - F(l_x) \leq \Lambda(l_x - l_y).$$

Semiconcavity

Theorem (C. Mou, Indiana Univ. Math. J. to appear)

Suppose that the nonlinearity G in (2) satisfies ($\bar{H}2$), and that $j(x,z)$ satisfies assumption ($\bar{H}1$). Then, if $u \in C^{0,1}(\mathbb{R}^n)$ is a viscosity solution of (2) and γ is sufficiently large, we have u is $\bar{\theta}$-semiconcave in $\mathbb{R}^n$.

Theorem (C. Mou, Indiana Univ. Math. J. to appear)

Suppose that $c_\alpha \geq \gamma$ *in* $\mathbb{R}^n$, *that the family* $\{j_\alpha(x,z)\}$ *satisfies assumption* $(\bar{H}1)$, *and that* $c_\alpha \in C^{1,\bar{\theta}-1}(\bar{\mathbb{R}}^n)$ *and* $\{f_\alpha\}$ *is uniformly* $\bar{\theta}$*-semiconvex. Suppose that*

$$\sup_{\alpha \in \mathcal{A}} \max\{|\sigma_\alpha(0)|, |b_\alpha(0)|\} < +\infty,$$

and

$$\sup_{\alpha \in \mathcal{A}} \max\{[\sigma_\alpha]_{0,1;\mathbb{R}^n}, [\sigma_\alpha]_{1,\bar{\theta}-1;\mathbb{R}^n}, [b_\alpha]_{0,1;\mathbb{R}^n}, [b_\alpha]_{1,\bar{\theta}-1;\mathbb{R}^n}, [f_\alpha]_{0,1,\mathbb{R}^n}\} < +\infty.$$

Then, if $u \in C^{0,1}(\mathbb{R}^n)$ *is a viscosity solution of (1) and* γ *is sufficiently large, we have* u *is* $\bar{\theta}$*-semiconcave in* $\mathbb{R}^n$.

Thank you for your attention!