A Local-Nonlocal Transmission Problem

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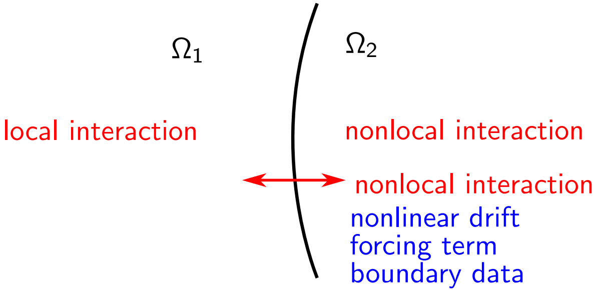
The Setting



Ω_1, Ω_2 : nice open sets partitioning $\mathbb{R}^n$.
Physical model based on surface quasigeostrophic equation, where Ω_1 is water (free lower boundary) and Ω_2 is land (fixed lower boundary).
Important: problem has an energy structure.

The PDE (in weak form)

u solves (P) on domain $U \times (0, T]$ if for all $\phi \in C_c^\infty(U \times (0, T))$,

$$\int \partial_t \phi u + \int_{\Omega_1} \nabla u \cdot \nabla \phi + \int_0^T \int_{\Omega_2 \times \Omega_2} \frac{[u(x,t) - u(y,t)][\phi(x,t) - \phi(y,t)]}{|x-y|^{n+2s}}$$

$$+ \nu \int_0^T \int_{\Omega_1 \times \Omega_2} \frac{[u(x,t) - u(y,t)][\phi(x,t) - \phi(y,t)]dxdy}{|x-y|^{n+2s}}$$

$$- \int ub \cdot \nabla \phi + f\phi = 0$$

Here $\nu > 0$, b is a divergence-free vector field. Physical case: $n = 2$, $s = 1/2$, $b = (-R_2 u, R_1 u)$, R_i are Riesz transforms.

The PDE (stationary)

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The PDE (stationary)

u solves (P) on domain U if for all $\phi \in C_c^\infty(U)$),

$$\begin{aligned}\int_{\Omega_1} \nabla u \cdot \nabla \phi + \int_{\Omega_2 \times \Omega_2} \frac{[u(x) - u(y)][\phi(x) - \phi(y)] dx dy}{|x - y|^{n+2s}} \\ + \nu \int_{\Omega_1 \times \Omega_2} \frac{[u(x) - u(y)][\phi(x) - \phi(y)] dx dy}{|x - y|^{n+2s}} \\ - \int ub \cdot \nabla \phi + f \phi = 0\end{aligned}$$

The PDE (simplified)

u solves (P) on domain U if for all $\phi \in C_c^\infty(U))$,

$$B_L[u,\phi] + B_N[u,\phi] = Blue$$

Goals

1. Do solutions exist?
2. Do they behave reasonably near the interface $\Gamma = \partial\Omega_1$ (e.g. are they continuous?)
3. How do they look like near Γ?
4. Why do I call this a transmission problem?

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1. Do solutions exist? Yes. $u \in H^1(\Omega_1) \cap H^s$
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Continuity

Solution: obtain an extra family of scale-invariant energy estimates on Ω_2.

Theorem

Assume u solves (P) on B_1, that $f \in L^\infty$, Γ is a Lipschitz graph, and one of:

1. $s \geq 1/2$ *and* $b \in L^p, p \geq \frac{n}{2s-1}, \operatorname{div} b = 0$
2. $s \geq 1/2$ *and* $b = (-R_2 u, R_1 u)$
3. $s \in (0,1)$, $b = 0$.

Then $u \in C^{0,\alpha}(B_{1/2})$.

Note: f can be in the appropriate L^p spaces instead.

Note: I have no parabolic version of this.

Structure of solutions, simplest case

Assume $\Gamma = \{x_n = 0\}$, $\Omega_1 = \{x_n < 0\}$, $b = 0$. How do solutions look like near Γ?

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Theorem

There is an explicit number $\alpha_0 = \alpha_0(\nu, s) \in ((2s-1)_+, 2s)$ and $M_0 = M_0(\nu, s)$ such that

1. $u \in C^{\alpha_0}(\bar{\Omega}_2)$ *(with $u_n = 0$ from that side if $\alpha_0 > 1$)*
2. $u \in C^{\alpha_0+2-2s}(\bar{\Omega}_1)$ *(with $u_n = 0$ from that side)*
3. $\lim_{t \searrow 0} \frac{u(x',-t)-u(x',0)}{t^{\alpha_0+2-2s}} = M_0 \lim_{t \searrow 0} \frac{u(x',t)-u(x',0)}{t^{\alpha_0}}$ *for all x'.*

Note: can obtain a whole asymptotic expansion for u.

Structure of solutions, general case

Assume now Γ smooth, b continuous. Can localize and flatten interface so that Γ is flat but now there are coefficients:

$$
\begin{aligned}
&\int_{\Omega_1} A\nabla u \cdot \nabla\phi \\
&+ \int_{\Omega_2\times\Omega_2} \frac{[u(x)-u(y)]a(x,y)[\phi(x)-\phi(y)]dxdy}{|x-y|^{n+2s}} \\
&+ \nu\int_{\Omega_1\times\Omega_2} \frac{[u(x)-u(y)]a(x,y)[\phi(x)-\phi(y)]dxdy}{|x-y|^{n+2s}} \\
&- \int ub\cdot\nabla\phi + f\phi = 0
\end{aligned}
$$

Structure of solutions, general case

Essentially everything still works if the *compatibility condition* is satisfied. This is a single algebraic relation between the averages and first moments of $a(x, y)$, the matrix A, and the normal vector to Γ. It asks that anisotropies are somehow matched appropriately.
Note: compatibility is conserved under (intelligent) interface flattening.
Note: if $s = 1/2$, then α_0, M_0 depend on $b(0) \cdot e_n$, the normal component of the drift.

Open problems

- Parabolic version (with time-dependent coefficients and/or critical drift)
- I can handle $\nu = 0$ (if $s \geq 1/2$). How about ν slightly negative?
- What if the $\Omega_1 \times \Omega_2$ term is higher-order than the $\Omega_2 \times \Omega_2$ term?

Thank you!