Uniqueness results for SDEs with jumps and Hölder continuous drift term

Enrico Priola

Torino (Italy)

Bedlewo

III Conference on Nonlocal Operators

27 June - 1 july 2016

Introduction

Consider the SDE

$$X_t = x + \int_0^t b\left(X_s\right) \mathrm{d}s + L_t, \qquad x \in \mathbb{R}^d, \;\; t \geqslant 0, \tag{1}$$

assuming

$b : \mathbb{R}^d \to \mathbb{R}^d$ is *bounded* and β-Hölder continuous, (i.e. $b \in C_b^\beta(\mathbb{R}^d, \mathbb{R}^d)$; $\|b\|_\beta = \|b\|_0 + [b]_\beta < +\infty$),

$$[b]_\beta = \sup_{x \neq y,\, x,y \in \mathbb{R}^d} \frac{|b(x) - b(y)|}{|x-y|^\beta}, \quad \beta \in (0,1).$$

$L = (L_t)$ is a non-degenerate d-dimensional Lévy process, $d \geqslant 1$, defined on $(\Omega, \mathcal{F}, \mathbb{P})$ (or more generally on $\mathcal{BS} = (\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$)

Stable type Lévy processes will be important in the sequel.
We can write

$$X_t(\omega) = x + \int_0^t b\left(X_s(\omega)\right) \mathrm{d}s + L_t(\omega), \qquad \omega \in \Omega.$$

Plan of the talk

We concentrate on

- **Analytic results and pathwise uniqueness for the SDE**

- **Davie's type uniqueness for the SDE**

The talk is based on the following papers:

- E. Priola, Pathwise uniqueness for singular SDEs driven by stable processes, Osaka Journal of Mathematics (2012)

- E. Priola, Stochastic flow for SDEs with jumps and irregular drift term, Banach Center Publications vol. 105, 193-210 (2015) (dedicated to prof. J. Zabczyk)

- E. Priola, Davie's type uniqueness for for a class of SDEs with jumps, Preprint Arxiv 2015.

I Definitions (Lévy process L)

For any $t \geqslant 0$, $L(t) = L_t$ is a measurable function (random variable) defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{R}^d$; moreover

- $L(0) = 0$, $\mathbb{P}$-a.s.;
- for $0 = t_0 < t_1 < t_2 < \ldots < t_n$, the increments

  $$L(t_1) - L(t_0), L(t_2) - L(t_1), \ldots, L(t_n) - L(t_{n-1}) \text{ are independent;}$$

- the random variable $L(t) - L(s)$, for $0 \leqslant s < t$, has the same law as $L(t - s)$;
- $\lim_{h \to 0} \mathbb{P}(|L(t+h) - L(t)| > \epsilon) = 0$, for any $\epsilon > 0$, $t \geqslant 0$;
- each trajectory $t \mapsto L(t, \omega) = L_t(\omega) \in \mathbb{R}^d$ is cadlag (right continuous with left limits) on $[0, +\infty)$, for ω $\mathbb{P}$-a.s.

Important examples are the Wiener process and the Poisson process.

If we define the σ-algebra $\mathcal{F}_t^L = \sigma(L_s,\ s \leqslant t)$ completed with $\mathbb{P}$-null sets we get that

$$(\Omega, \mathcal{F}, (\mathcal{F}_t^L)_{t \geqslant 0}, \mathbb{P}).$$

is a stochastic basis.

II Definitions

More generally, on a fixed stochastic basis

$$\mathcal{BS} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P}).$$

a d-dimensional process L defined and adapted on $\mathcal{BS}$ is a Lévy process if it is continuous in probability, it has stationary increments, càdlàg paths (i.e., $\mathbb{P}$-a.s., each mapping $t \mapsto L_t(\omega)$ is càdlàg from $[0, \infty)$ into $\mathbb{R}^d$) $L_t - L_s$ is independent of $\mathcal{F}_s$, $0 \leqslant s \leqslant t$, and $L_0 = 0$, $\mathbb{P}$-a.s..

Pathwise uniqueness : Consider two d-dimensional processes $X = (X_t)$ and $Y = (Y_t)$ defined and $(\mathcal{F}_t)$–adapted on $\mathcal{BS}$ (i.e. $X_t, Y_t : (\Omega, \mathcal{F}_t) \to \mathbb{R}^d$ are measurable for any t).

If X and Y solve (1), for any $\omega \in \Omega$, $\mathbb{P}$-a.s., then we have, $\mathbb{P}$-a.s.,

$$X_t = Y_t. \quad t \geqslant 0.$$

Strong existence : A solution X defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, L)$ starting from x at time 0 is called **strong solution** if it is $(\mathcal{F}_t^L)$–adapted

($\mathcal{F}_t^L = \sigma(L_s,\ s \leqslant t)$ completed with $\mathbb{P}$-null sets)

Pathwise uniqueness for singular SDEs when $L = W$ (Wiener case)

(the deterministic equation with $L = 0$ is not well-posed)

- A.K. Zvonkin : Mat. Sb. (N.S.) (1974) [$b \in L^\infty(\mathbb{R})$, i.e., $d = 1$]

- A.J. Veretennikov : Mat. Sb., (N.S.) (1980) [$b \in L^\infty(\mathbb{R}^d)$ (for any $d \geqslant 1$)].

- I. Gyöngy and T. Martinez : Czechoslovak Math. J. (2001) [$b \in L^{2d+2}_{loc}(\mathbb{R}^d)$] plus growth conditions]

- N.V. Krylov and M. Röckner : Probab. Theory Relat. Fields (2005) [$b \in L^p_{loc}(\mathbb{R}^d)$ when $p > d$, generalizing Portenko(1982)]

- X. Zhang : Strong solutions of SDEs with singular drift and Sobolev diffusion coefficients. Stoch. Proc. and Appl. (2005).

- E. Fedrizzi, F. Flandoli, Hölder flow and differentiability for SDEs with nonregular drift. Stoch. Anal. Appl. 31 (2013)

- L. Beck, F. Flandoli, M. Gubinelli, M. Maurelli, Stochastic ODEs and stochastic linear PDEs with critical drift: regularity, duality and uniqueness, Preprint Arxiv.

- .....

Pathwise uniqueness when $L \neq W$

Recall that a d-dimensional process Lévy process L can be characterized by its **symbol (or exponent)** $\psi : \mathbb{R}^d \to \mathbb{R}$,

$$\mathbb{E}[e^{i\langle L_t, h\rangle}] = e^{-t\psi(h)}, \quad t \geqslant 0,\ h \in \mathbb{R}^d,$$

(here $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^d$); see [Sato 1999] and [Applebaum 2004].

When $\mathbb{R}^d = \mathbb{R}$, L is a **symmetric α-stable Lévy process** if

$$\psi(h) = c_\alpha |h|^\alpha, \quad h \in \mathbb{R},$$

for some $\alpha \in (0, 2)$.

H. Tanaka, M. Tsuchiya and S. Watanabe proved in J. Math. Kyoto Univ. (1974) when $d = 1$:

if $\alpha < 1$ and $b \in C_b^\beta(\mathbb{R})$ with $\alpha + \beta < 1$ **pathwise uniqueness may fail.**

(in contrast with the case of the Wiener process)

Some recent papers on strong existence and uniqueness when $L \neq W$

- E. Priola, Pathwise uniqueness for singular SDEs driven by stable processes, Osaka Journal of Mathematics (2012)

- X. Zhang, Stochastic differential equations with Sobolev drifts and driven by α-stable processes, Ann. Inst. H. Poincaré Probab. Statist. (2013)

- S. Haadem and F. Proske, On the construction and Malliavin differentiability of solutions of Lévy noise driven SDE's with singular coefficients, JFA (2014)

- E. Priola, Stochastic flow for SDEs with jumps and irregular drift term, Banach Center Publications (2015)

- Z. Q. Chen, R. Song, X. Zhang, Stochastic flows for Lévy processes with Hölder drifts, Preprint Arxiv 2015

Two examples of Lévy processes to be considered

We concentrate on *pure-jump Lévy process L without drift term,* i.e., we assume

$$\mathbb{E}[e^{i\langle L_t,u\rangle}] = e^{-t\psi(u)},$$

$$\psi(u) = -\int_{\mathbb{R}^d} \Big(e^{i\langle u,y\rangle} - 1 - i\langle u,y\rangle\, 1_{\{|x|\leqslant 1\}}(y)\Big)\nu(dy),$$

$u \in \mathbb{R}^d$, $t \geqslant 0$. The measure ν is also called **the Lévy measure** of L; ν is a σ-finite measure on $\mathbb{R}^d$ with $\nu(\{0\}) = 0$ and

$$\int_{\mathbb{R}^d} (1 \wedge |y|^2)\, \nu(dy) < \infty;$$

Example 1. $L = (L_t)$ is a d-dimensional Lévy process, $d \geqslant 1$, such that

$$\mathbb{E}[e^{i\langle L_t,h\rangle}] = e^{-c_\alpha t|h|^\alpha}, \quad h \in \mathbb{R}^d, \quad t \geqslant 0, \quad \alpha \in (0,2);$$

L is the standard (symmetric and rotationally invariant) α-stable process.

The generator of L is $\mathcal{L} = -(-\triangle)^{\alpha/2}$; for any $x \in \mathbb{R}^d, f \in C_0^\infty(\mathbb{R}^d)$,

$$-(-\triangle)^{\alpha/2} f(x) = \int_{\mathbb{R}^d} \big(f(x+y) - f(x) - 1_{\{|y|\leqslant 1\}}\, \langle y, Df(x)\rangle\big) \frac{\tilde{c}_\alpha}{|y|^{d+\alpha}} dy. \quad (2)$$

The measure ν with density $\frac{\tilde{c}_\alpha}{|y|^{d+\alpha}}$ is the Lévy measure of the process L.

Example 2. L is a Lévy process such that

$$\mathbb{E}[e^{i\langle L_t,h\rangle}] = e^{-c_\alpha t(|h_1|^\alpha+\ldots+|h_d|^\alpha)}, \quad h\in\mathbb{R}^d, \quad t\geqslant 0, \quad \alpha\in(0,2).$$

In this case $L=(L_t^1,\ldots,L_t^d)$, where $L^1,\ldots,L^d$ are independent one-dimensional symmetric α-stable processes. The generator of L is

$$\mathcal{L} = -\sum_{k=1}^{d}(-\partial^2_{x_k x_k})^{\alpha/2},$$

i.e., for any $x\in\mathbb{R}^d, f\in C_0^\infty(\mathbb{R}^d)$,

$$\mathcal{L}f(x) = \sum_{k=1}^{d}\int_{\mathbb{R}}[f(x+se_k)-f(x)-1_{\{|s|\leqslant 1\}}\, s\,\partial_{x_k}f(x)]\,\frac{c_\alpha}{|s|^{1+\alpha}}ds.$$

Martingale problems for SDEs with multiplicative noise driven by $(L_t^1,\ldots,L_t^d)$ *have been studied in [Bass-Chen 2006].* □

In the sequel we will consider the **Blumenthal-Getoor index** $\alpha_0=\alpha_0(\nu)$:

$$\alpha_0 = \inf\left\{\sigma>0 \;:\; \int_{\{|x|\leqslant 1\}}|y|^\sigma\nu(dy)<\infty\right\}; \tag{3}$$

we always have $\alpha_0\in[0,2]$. In the sequel we require $\alpha_0\in(0,2)$.

Pathwise uniqueness by regular solutions to Kolmogorov equation

Theorem (P. 2015)

Let L be a pure Lévy process such that $\alpha_0 = \alpha_0(\nu) \in (0,2)$. *Suppose that, for some* $\lambda > 0$, *there exists* $u = u_\lambda \in C_b^{1+\gamma}(\mathbb{R}^d, \mathbb{R}^d)$, $\gamma \in]0,1]$ *and* $2\gamma > \alpha_0$, *which solves*

$$\lambda u(x) - \mathcal{L}u(x) - Du(x)\, b(x) = b(x), \quad x \in \mathbb{R}^d, \tag{4}$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ *is given in the SDE,* $\mathcal{L}$ ***is the generator of*** L *and* $\lambda > 0$;

Eq. (4) *is componentwise, i.e.,* $u : \mathbb{R}^d \to \mathbb{R}^d$ *and, with* $\mathcal{L}_b = \mathcal{L} + b(x) \cdot D$,

$$\lambda u_k(x) - \mathcal{L}_b u_k(x) = b_k(x), \quad k = 1, \ldots, d, \tag{5}$$

Moreover, assume $\|Du_\lambda\|_0 < 1/3$.

Then, for any $x \in \mathbb{R}^d$, *strong existence and pathwise uniqueness hold for the SDE.*

$$\mathcal{L}g(x) = \int_{\mathbb{R}^d} \big(g(x+y) - g(x) - 1_{\{|y| \leqslant 1\}} \langle y, Dg(x) \rangle\big)\, \nu(dy), \; g \in C_0^\infty(\mathbb{R}^d).$$

The proof uses an **Itô-Tanaka trick** like in [Flandoli-Gubinelli-P. 2010] for the case $L = W$; this is a variant of Zvonkin's method of [Veretennikov 1980].

The Itô-Tanaka trick when $W = L$

To simplify we consider $\mathbb{R}^d = \mathbb{R}$. We write

$$X_t - x - W_t = \int_0^t b(X_s)ds.$$

Now if v is a "regular" solution of

$$\lambda v - Lv = b \quad \text{on } \mathbb{R}, \quad \lambda > 0,$$

$L = \frac{1}{2}\frac{d^2}{dx^2} + b(x) \cdot \frac{d}{dx}$ then by Itô's formula:

$$v(X_t) = v(x) + \int_0^t v'(X_s)dW_s + \int_0^t Lv(X_s)ds$$

and so

$$v(X_t) = v(x) + \int_0^t v'(X_s)dW_s + \int_0^t (\lambda v(X_s) - b(X_s))ds$$

and

$$X_t + v(X_t) = x + v(x) + W_t + \int_0^t v'(X_s)dW_s + \lambda \int_0^t v(X_s)ds$$

allows to get uniqueness.

Simple comments on the Itô-Tanaka trick

When we say by Itô's formula:

$$v(X_t) = v(x) + \int_0^t v'(X_s)dW_s + \int_0^t Lv(X_s)ds$$

we have to define the stochastic Itô integral

$$\int_0^t v'(X_s)dW_s$$

(trajectories $s \mapsto W_s(\omega)$ are not of bounded variation). ,

One uses that, for any $t \geqslant 0$,

$$X_t : (\Omega, \mathcal{F}_t) \to \mathbb{R}^d \quad \text{is measurable}$$

This implies that, X_t is independent of $W_{t+h} - W_t$, for any $t, h \geqslant 0$.
We can then define

$$\int_0^1 v'(X_s)dW_s$$

as **limit in** $L^2(\Omega)$ of suitable Riemann sums like

$$\sum_{t_k, t_{k+1} \in \pi} v'(X_{t_k})\,(W_{t_{k+1}} - W_{t_k})$$

as the mesh of partition $\pi = \{0 = t_0 < .... < t_n = 1\}$ tends to 0.

Hypothesis on the Lévy process L to solve the previous Kolmogorov equation

(HK) We require that $\alpha_0 \in (0,2)$ and moreover the convolution semigroup (P_t),

$$P_t f(x) = \mathbb{E}[f(x+L_t)], \quad f \in C_b(\mathbb{R}^d),\ t \geqslant 0,\ x \in \mathbb{R}^d,$$

verifies: $P_t(C_b(\mathbb{R}^d)) \subset C_b^1(\mathbb{R}^d)$, $t > 0$, and, moreover, there exists $c_{\alpha_0} = c_{\alpha_0}(\nu) > 0$ such that

$$\sup_{x\in\mathbb{R}^d} |DP_t f(x)| \leqslant c_{\alpha_0}\, t^{-\frac{1}{\alpha_0}} \cdot \sup_{x\in\mathbb{R}^d} |f(x)|, \quad t \in (0,1],\ f \in C_b(\mathbb{R}^d). \tag{6}$$

Examples. *Non-degenerate symmetric stable processes, relativistic and truncated stable processes verifies* (HK).

On the previous gradient estimates

There is criterion which is based on [Schilling, Sztonyk and Wang, 2012]:

Let L be a pure-jump Lévy process. A sufficient condition in order that **gradient estimates** (6) holds when α_0 replaced by $\gamma \in (0, 2)$ is the following one:

the Lévy measure ν of L verifies:

$$\nu(B) \geqslant \nu_1(B), B \in \mathcal{B}(\mathbb{R}^d),$$

where ν_1 is a Lévy measure on $\mathbb{R}^d$ such that its corresponding symbol

$$\psi_1(h) = -\int_{\mathbb{R}^d} \left(e^{i\langle h,y\rangle} - 1 - i\langle h, y\rangle\, 1_{\{|y| \leqslant 1\}}\,(y)\right)\nu_1(dy),$$

satisfies, for some positive constants c_1, c_2 and M,

$$c_1|x|^\gamma \leqslant Re\, \psi(x) \leqslant c_2|x|^\gamma, \quad \text{when } |x| > M.$$

Solvability of the Kolmogorov equation when $\alpha_0 \geqslant 1$

We consider $b \in C_b^{\beta}(\mathbb{R}^d, \mathbb{R}^d)$.

Theorem (P. 2015)

Assume (HK). Let $\alpha_0 \geqslant 1$ and $\beta \in (0,1)$ be such that $1 < \alpha_0 + \beta < 2$. Then, for any $\lambda > 0$, $g \in C_b^{\beta}(\mathbb{R}^d)$, there exists a unique solution $u = u_\lambda \in C_b^{\alpha_0+\beta}(\mathbb{R}^d)$ to

$$\lambda u - \mathcal{L}u - b \cdot Du = g$$

on $\mathbb{R}^d$. Moreover, for any $\omega > 0$, there exists $c = c(\omega)$, independent of g and u, such that

$$\lambda \|u\|_0 + [Du]_{\alpha_0+\beta-1} \leqslant c\|g\|_{\beta}, \quad \lambda \geqslant \omega. \tag{7}$$

Finally, we have $\lim_{\lambda\to\infty} \|Du_\lambda\|_0 = 0$.

Remark Note that $\gamma = \alpha_0 + \beta - 1$. Hence the condition $2\gamma > \alpha_0$ in the theorem on pathwise uniqueness becomes

$$\beta > 1 - \alpha_0/2.$$

Remark that the last assertion follows from (23). Indeed we obtain, for $\lambda \geqslant \omega$,

$$\|Du_\lambda\|_0 \leqslant N[Du_\lambda]_{\alpha_0+\beta-1}^{\frac{1}{\alpha_0+\beta}} \|u_\lambda\|_0^{1-\frac{1}{\alpha_0+\beta}} \leqslant N\tilde{c}\,\lambda^{-\frac{\alpha_0+\beta-1}{\alpha_0+\beta}} \|g\|_\beta,$$

where $\tilde{c} = \tilde{c}(\omega)$ and $N = N(d, \alpha_0, \beta)$.

The proof is based on a-priori estimates which allow to use the well known continuity method.

Such estimates are not difficult when $\boxed{\alpha_0 > 1}$. Indeed in the relevant case $\mathcal{L} = -\sum_{k=1}^{d}(-\partial^2_{x_k x_k})^{\alpha_0/2}$, the term

$$\boxed{b \cdot Du \quad \text{is "of lower order" with respect to } -\sum_{k=1}^{d}(-\partial^2_{x_k x_k})^{\alpha_0/2}u.}$$

The critical case $\alpha_0 = 1$ requires to use a localization procedure to get apriori-estimates.

This is based on the fact that in the first theorem (where $b = b_0$) the Schauder constant is independent on b_0.

Solvability when $0 < \alpha_0 < 1$: only the fractional Laplacian

I can only treat the case when $\mathcal{L}$ is the fractional Laplacian **using apriori estimates of [Silvestre 2012]**

Theorem (P. 2015)

Assume the $\mathcal{L}$ is the fractional Laplacian.
Let $0 < \alpha_0 < 1$ *and* $\beta \in (0,1)$ *be such that* $1 < \alpha_0 + \beta < 2$. *Then, for any* $\lambda > 0$, $g \in C_b^{\beta}(\mathbb{R}^d)$, *there exists a unique solution* $u = u_\lambda \in C_b^{\alpha_0+\beta}(\mathbb{R}^d)$ *to*

$$\lambda u - \mathcal{L}u - b \cdot Du = g$$

on $\mathbb{R}^d$. *Moreover, for any* $\omega > 0$, *there exists* $c = c(\omega)$, *independent of* g *and* u, *such that*

$$\lambda \|u\|_0 + [Du]_{\alpha_0+\beta-1} \leqslant c\|g\|_\beta, \quad \lambda \geqslant \omega. \tag{8}$$

Finally, we have $\lim_{\lambda\to\infty} \|Du_\lambda\|_0 = 0$.

Silvestre starts with the so called **extension property** (see [Caffarelli-Silvestre2006] and [Molchanov-Ostrovskii1968])

$$\begin{cases} \operatorname{div}\big(y^{1-\alpha_0} Du(x,y)\big) = 0 \quad \text{on } \ \mathbb{R}^d \times \mathbb{R}_+ \\ \\ u(x,0) = f(x), \quad x \in \mathbb{R}^n, \end{cases}$$

$-(-\triangle)^{\alpha_0/2} f(x) = c \lim_{y\to 0^+} y^{1-\alpha_0} \partial_y u(x,y).$

Due to the *lack of extension property* it is not clear if for $\boxed{\alpha_0, \beta \in (0,1)}$, $\alpha_0 + \beta > 1$,

$$f,\ b \in C_b^{\beta} \implies v \in C_b^{\alpha_0+\beta}$$

even for

$$v(x_1,x_2) + (-\partial^2_{x_1 x_1})^{\alpha_0/2} v(x_1,x_2) + (-\partial^2_{x_2 x_2})^{\alpha_0/2} v(x_1,x_2) - b(x_1,x_2) \cdot Dv(x_1,x_2)$$

$$= f(x_1,x_2) \quad \text{on } \mathbb{R}^2.$$

However a pathwise uniqueness result holds when $\alpha_0 \in (2/3, 1)$; see [Chen-Song-Zhang 2015].

Pathwise uniqueness (full result)

Theorem (P. 2015)

Let L be a pure Lévy process. Suppose that L satisfies (HK) if $\alpha_0 = \alpha_0(\nu) \in [1,2)$*. If* $\alpha_0 \in (0,1)$ *then L is rotationally invariant and symmetric. Let* $\beta > 1 - \alpha_0/2$*. Then on* $(\Omega, \mathcal{F}, P)$*, for* $x \in \mathbb{R}^d$*, there exists a pathwise unique strong solution* $(X_t^x)_{t \geqslant 0}$.

Let now

$$dX_t = b(X_t)dt + dL_t, \quad s \leqslant t \leqslant T, \quad X_s = x, \quad s \in [0,T],\ x \in \mathbb{R}^d. \tag{9}$$

Here $\mathcal{F}_{s,t}^L$ **is the completion of the** σ**-algebra generated by** $L_r - L_s$**,** $r \in [s,t]$**.** We also set $\mathcal{F}_{0,t}^L = \mathcal{F}_t^L$. Strong solutions to (9) are $(\mathcal{F}_{s,t}^L)$-adapted.

Theorem (P. 2015)

Under the same hypotheses as before let $T > 0$ *and* $s \in [0,T]$*. Then, for any* $x \in \mathbb{R}^d$*, there exists a pathwise unique strong solution* $\tilde{X}^{s,x} = (\tilde{X}_t^{s,x})_{t \in [0,T]}$ *to (9) on* $(\Omega, \mathcal{F}, (\mathcal{F}_{s,t}^L), \mathbb{P})$ *(we set* $\tilde{X}_t^{s,x} = x$ *for* $t \leqslant s$*). Moreover if* $U^{s,x}$ *and* $U^{s,y}$ *are two strong solutions on* $[0,T]$ *defined on* $(\Omega, \mathcal{F}, (\mathcal{F}_{s,t}^L), P)$*, then we have, for* $p \geqslant 2$*,*

$$\sup_{s \in [0,T]} \mathbb{E}[\sup_{s \leqslant t \leqslant T} |U_t^{s,x} - U_t^{s,y}|^p] \leqslant C(T,p,d,b,\alpha_0)\, |x-y|^p, \quad x, y \in \mathbb{R}^d.$$

A uniqueness result by Davie

A.M. Davie [Int. Math. Res. Not. 2007]: a multidimensional stochastic equation

$$dX_t = b(t, X_t)\, dt + dW_t, \quad X_0 = x,$$

driven by a Wiener process $W = (W_t)$ with a coefficient b which is only **bounded and measurable** on $[0, T] \times \mathbb{R}^d$ has a unique solution for almost all choices of the driving Wiener path

$$t \mapsto W_t(\omega)$$

(NO Ito's solutions !)

Theorem (Davie 07)

There exists an event $\Omega' \in \mathcal{F}$ with $P(\Omega') = 1$ such that for any $\omega \in \Omega'$, $x \in \mathbb{R}^d$, the integral equation

$$f(t) = x + \int_0^t b\,(r, f(r) + W_r(\omega))\, dr, \quad t \in [0, T],$$

has exactly one solution f in $C([0, T]; \mathbb{R}^d)$.

On Davie's type uniqueness in the Lévy case

We consider a similar problem when the Wiener process W is replaced by a pure jump Lévy process $L = (L_t)$ as the ones considered before, for which pathwise uniqueness hold.

The problem seems to be open even in dimension one, for instance,

$$dX_t = (\sqrt{|X_t|} \wedge 1)\, dt + dL_t, \quad X_0 = x \in \mathbb{R},$$

with a *symmetric α-stable process $L = (L_t)$.*

REMARK The proof in [Davie07] is self-contained but very technical; it relies on explicit computations with Gaussian kernels. An alternative approach to the Davie uniqueness result has been proposed in

A. V. Shaposhnikov, *Some remarks on Davie's uniqueness theorem*, Arxiv 2014

It uses the flow property of SDEs driven by the Wiener process.

Beside [Davie07] our work has been also inspired by [Shaposhnikov14].

General hypotheses on L and b for Davie's uniqueness

(H1) The drift $b(t,x)$ is bounded measurable and β**-Hölder continuous** in the space variable, uniformly in t.

The d-dimensional *Lévy process* $L = (L_t) = (L_t)_{t \geqslant 0}$, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with generating triplet $(\nu, Q, 0)$ verifies:

(H2)

$$\mathbb{E}|L_1|^\theta < \infty,$$

for some $\theta > 0$.

(H3) (i) For any $s \in [0,T]$ and $x \in \mathbb{R}^d$ on $(\Omega, \mathcal{F}, P)$ there exists a strong solution $(U_t^{s,x})_{t\in[0,T]}$ to

$$dX_t = b(t, X_t)dt + dL_t, \quad s \leqslant t \leqslant T, \quad X_s = x,$$

(ii) Let $s \in [0,T]$. Given two strong solutions $(U_t^{s,x})_{t\in[0,T]}$ and $(U_t^{s,y})_{t\in[0,T]}$ defined on $(\Omega, \mathcal{F}, P)$ which solve the SDE with respect to L and b we have

$$\sup_{s\in[0,T]} E\Big[\sup_{s\leqslant t\leqslant T} |U_t^{s,x} - U_t^{s,y}|^p\Big] \leqslant C(T)\,|x-y|^p, \quad x, y \in \mathbb{R}^d, \ p \geqslant 2,$$

with $C(T) = C\big((\nu, Q, 0), \|b\|_{\beta,T}, d, \beta, p, T\big) > 0$ **independent of** s, x **and** y. □

A Davie's type result in the Lévy case

Theorem (P. Arxiv 2015)

Let us consider a Levy process which satisfies (H1) - (H3) and consider $b \in L^\infty(0,T; C_b^\beta(\mathbb{R}^d;\mathbb{R}^d))$, $\beta \in [0,1]$.

There exists an event $\Omega' \in \mathcal{F}$ *with* $P(\Omega') = 1$ *such that for any* $\omega \in \Omega'$, $x \in \mathbb{R}^d$, *the integral equation*

$$f(t) = x + \int_0^t b\,(r, f(r) + L_r(\omega))\,dr, \quad t \in [0,T],$$

has exactly one solution f *in* $C([0,T];\mathbb{R}^d)$.

The result will be obtained by using a suitable strong solution $\phi(s,t,x,\omega)$

One idea about the proof: if one has a "nice C_0*-semigroup* e^{tA}*" then one can prove uniqueness for*

$$\begin{cases} u'(t) = Au(t), \\ u(0) = x \end{cases}$$

by $\frac{d}{ds}(e^{(t-s)A}u(s)) = 0$ *which implies* $u(t) = e^{tA}x$.

Construction of a particular strong solution ϕ

To prove the theorem we will construct a function $\phi(s,t,x,\omega)$,

$$\phi : [0,T] \times [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d,$$

which is $\mathcal{B}([0,T] \times [0,T] \times \mathbb{R}^d) \times \mathcal{F}$-measurable and such that

$$\big(\phi(s,t,x,\cdot)\big)_{t\in[0,T]}$$

is a strong solution of

$$dX_t = b(t, X_t)dt + dL_t, \quad s \leqslant t \leqslant T, \quad X_s = x. \tag{10}$$

Moreover, there exists an almost sure event Ω' such that the next assertions on ϕ holds for any $\omega \in \Omega'$.

(In the sequel to deal with ϕ we cannot use directly results like the Kolmogorov test (if $L \neq W$) due to the cádlág dependence on s and t)

Assertions on ϕ for any $\omega \in \Omega'$

(i) For any $x \in \mathbb{R}^d$, the mapping: $s \mapsto \phi(s,t,x,\omega)$ is **càdlàg** on $[0,T]$ (uniformly in t and x),

i.e., let $s \in (0,T)$ and consider sequences (s_k) and (s_n) such that $s_k \to s^-$ and $s_n \to s^+$; we have, for any $M > 0$,

$$\lim_{n\to\infty} \sup_{|x|\leqslant M} \sup_{t\in[0,T]} |\phi(s_n,t,x,\omega) - \phi(s,t,x,\omega)| = 0, \tag{11}$$

$$\lim_{k\to\infty} \sup_{|x|\leqslant M} \sup_{t\in[0,T]} |\phi(s_k,t,x,\omega) - \phi(s-,t,x,\omega)| = 0$$

(similar conditions hold when $s = 0$ and $s = T$).

(ii) For any $x \in \mathbb{R}^d$, $s \in [0,T]$, $\phi(s,t,x,\omega) = x$ if $0 \leqslant t \leqslant s$, and

$$\phi(s,t,x,\omega) = x + \int_s^t b\,(r,\phi(s,r,x,\omega))\,dr + L_t(\omega) - L_s(\omega), \quad t \in [s,T]. \tag{12}$$

(iii) For any $s \in [0,T]$, the function $x \mapsto \phi(s,t,x,\omega)$ is **continuous** in x uniformly in t. Moreover

Other conditions on ϕ

Moreover, for any integer $n > 2d$, there exists a $\mathcal{B}([0,T]) \times \mathcal{F}$-measurable function $V_n : [0,T] \times \Omega \to \mathbb{R}_+$ such that $\int_0^T V_n(s,\omega)ds < \infty$ and

$$\sup_{t\in[0,T]} |\phi(s,t,x,\omega) - \phi(s,t,y,\omega)| \tag{13}$$

$$\leqslant V_n(s,\omega)\,|x-y|^{\frac{n-2d}{n}}\,[(|x| \vee |y|)^{\frac{2d+1}{n}} \vee 1], \quad x,y \in \mathbb{R}^d,\ n > 2d,\ s \in [0,T].$$

(iv) For any $0 \leqslant s < r \leqslant t \leqslant T$, $x \in \mathbb{R}^d$, we have

$$\phi(s,t,x,\omega) = \phi(r,t,\phi(s,r,x,\omega),\omega). \tag{14}$$

(v) Let $s_0 \in [0,T[$, $\tau = \tau(\omega) \in (s_0,T]$ and $x \in \mathbb{R}^d$. If a càdlàg function $g : [s_0,\tau[\to \mathbb{R}^d$ solves the integral equation

$$g(t) = x + \int_{s_0}^t b\,(r, g(r))\,dr + L_t(\omega) - L_{s_0}(\omega), \quad t \in [s_0,\tau[, \tag{15}$$

then we have $g(r) = \phi(s_0,r,x,\omega)$, for $r \in [s_0,\tau[$.

Some ideas on the proof of (i)-(v)

I step. Let $s \in [0,T]$, $x \in \mathbb{R}^d$. We start with a strong solution $(\tilde{X}_t^{s,x})_{t\in[0,T]}$ defined on $(\Omega, \mathcal{F}, P)$ and introduce the d-dimensional process $\tilde{Y}^{s,x} = (\tilde{Y}_t^{s,x})_{t\in[0,T]}$,

$$\tilde{Y}_t^{s,x} = \tilde{X}_t^{x,s} - (L_t - L_s).$$

On some almost sure event $\Omega_{s,x}$ (independent of t) we have

$$\tilde{Y}_t^{s,x} = x + \int_s^t b(r, \tilde{Y}_r^{s,x} + (L_r - L_s))dr, \quad t \geqslant s,$$

and $\tilde{Y}_t^{s,x} = x$ on Ω if $t \leqslant s$. It follows that $(\tilde{Y}_t^{s,x})_{t\in[0,T]}$ has **continuous paths**.

II step. **Let s be fixed.** Using [Imkeller and Scheutzow 99] we show, *passing to a modification $Y^{s,x}$*, that, for any $p > 2d$, $\exists$ r.v.

$$U_{s,p} \geqslant 0$$

such that, for any $\omega \in \Omega$, $x, y \in \mathbb{R}^d$,

$$\|Y^{s,x}(\omega) - Y^{s,y}(\omega)\|_{G_0} \leqslant U_{s,p}(\omega)\,[(|x| \vee |y|)^{\frac{2d+1}{p}} \vee 1]\,|x-y|^{1-2d/p},$$

where $G_0 = C([0,T];\mathbb{R}^d)$.

Since for some almost sure event $\Omega'_{s,x}$ we have $Y_t^{s,x}(\omega) = \tilde{Y}_t^{s,x}(\omega)$, $\omega \in \Omega'_{s,x}$, $t \in [0, T]$, we obtain on $\Omega'_{s,x}$:

$$Y_t^{s,x} = x + \int_s^t b(r, Y_r^{s,x} + (L_r - L_s))dr, \quad t \geqslant s.$$

For $s, t \in [0, T]$, $x \in \mathbb{R}^d$, consider the r.v.

$$J_t^{s,x} = x + \int_s^t b(r, Y_r^{s,x} + (L_r - L_s))dr, \quad t \geqslant s,$$

$J_t^{s,x} = x$, $t < s$, $x \in \mathbb{R}^d$. **Let us fix** $s \in [0, T]$. We have

$$Y_t^{s,x}(\omega) = J_t^{s,x}(\omega), \quad \forall t \in [0, T],\ x \in \mathbb{R}^d \cap \mathbb{Q}^d \ \text{ if } \omega \in \Omega'_s = \bigcap_{x \in \mathbb{R}^d \cap \mathbb{Q}^d} \Omega'_{s,x}.$$

Since for fixed $s, t \in [0, T]$ the mapping: $x \mapsto J_t^{s,x}(\omega)$ is continuous from $\mathbb{R}^d$ into $\mathbb{R}^d$, for any $\omega \in \Omega$, we have

$$Y_t^{s,x}(\omega) = J_t^{s,x}(\omega), \quad t \in [0, T],\ x \in \mathbb{R}^d.$$

This shows that on some P-a.s. event Ω'_s (independent of t and x) we have

$$Y_t^{s,x} = x + \int_s^t b(r, Y_r^{s,x} + (L_r - L_s))dr, \quad t \geqslant s,\ x \in \mathbb{R}^d$$

III step. Consider the process $Y = (Y^s)$ with values in

$$C(\mathbb{R}^d; G_0).$$

This space is a separable complete metric space with the metric

$$d_0(f,g) = \sum_{N \geqslant 1} \frac{1}{2^N} \frac{\sup_{|x| \leqslant N} \|f(x) - g(x)\|_{G_0}}{1 + \sup_{|x| \leqslant N} \|f(x) - g(x)\|_{G_0}}, \quad f, g \in C(\mathbb{R}^d; G_0).$$

which corresponds to the compact-open topology.

Using a corollary of [Bezandry, X. Fernique 90]

Corollary (Bezardy-Fernique)

Let $X = (X_t)_{t \in [0,T]}$ be a stochastically continuous process with values in a complete metric space (S, d). A sufficient condition in order that X has a càdlàg modification is the following one: there exists $q > 1/2$ and $r > 0$ such that, for any $0 \leqslant s < t < u \leqslant T$, we have

$$E\left[d(X_s, X_t)^q \cdot d(X_t, X_u)^q\right] \leqslant C|u - s|^{1+r}.$$

we prove that there exists a modification $Z = (Z^s)$ of Y with càdlàg paths.

This result seems to be new even when b is Lipschitz and $d = 1$.

IV step. We define

$$\phi(s,t,x,\omega) = Z_t^{s,x}(\omega) + L_t(\omega) - L_s(\omega), \quad \text{if } s \leqslant t.$$

To prove the uniqueness part let $\omega \in \Omega'$ be fixed and let

$$g : [s_0, \tau[\to \mathbb{R}^d \quad \text{be a solution}$$

to the integral equation (15) corresponding to ω.
Let us fix $t \in (s_0, \tau)$.

We introduce an auxiliary function $f : [s_0, t] \to \mathbb{R}^d$

$$f(s) = \phi(s,t,g(s),\omega), \quad s \in [s_0, t]. \tag{16}$$

We show that f is constant on $[s_0, t]$.

Once this is proved we can deduce that

$$g(t) = f(t) = f(s_0) = \phi(s_0, t, x, \omega).$$

An example

One can consider

$$dX_t = \sqrt{|X_t|}\,dt + dL_t, \quad X_0 = x \in \mathbb{R}, \tag{17}$$

with a symmetric α-stable process $L = (L_t)$, $\alpha > 1$, and prove that **for almost all $\omega \in \Omega$ there exists one solution.**

Moreover, by a localization procedure, working ω by ω, one can construct the strong solution to (17).

Thank you

Some ideas about the proof: **pathwise uniqueness**

Poisson Random measures. The Poisson random measure N associated to L is

$$N((0,t] \times U) = \sum_{0<s\leqslant t} 1_U(\triangle L_s) = \sharp\{0 < s \leqslant t \ : \ \triangle L_s \in U\},$$

for any $U \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$, $t > 0$.
Here $\triangle L_s = L_s - L_{s-}$ is the jump size of L at time $s > 0$. The compensated Poisson random measure $\tilde{N}$ is defined by

$$\tilde{N}((0,t] \times U) = N((0,t] \times U) - t\nu(U),$$

where ν is the Lévy measure.
Recall

$$L_t = \int_0^t \int_{\{|x|\leqslant 1\}} x\tilde{N}(ds,dx) + \int_0^t \int_{\{|x|>1\}} xN(ds,dx), \quad t \geqslant 0, \tag{18}$$

$\int_0^t \int_{\{|x|\leqslant 1\}} x\tilde{N}(ds,dx)$ is the compensated sum of small jumps.

I Step. Let $X = X_t$ be a solution.
We will apply Itô's formula with $u_\lambda = u \in C_b^{1+\gamma}(\mathbb{R}^d;\mathbb{R}^d)$. Indeed

Itô formula holds under such regularity since $1+\gamma > \alpha_0$ (recall that $2\gamma > \alpha_0$)

Recall that (componentwise) on $\mathbb{R}^d$:

$$\lambda u - \mathcal{L}u - Du \cdot b = b. \tag{19}$$

Since there exists a solution $u = u_\lambda \in C_b^{1+\gamma}(\mathbb{R}^d, \mathbb{R}^d)$ to (19) (with $\gamma \in [0,1]$) such that $\boxed{1 + \gamma > \alpha_0}$ then by Itô's formula

$$u(X_t) - u(x) = \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} [u(X_{s-} + x) - u(X_{s-})] \tilde{N}(ds, dx) + \int_0^t \big(\mathcal{L}u(X_s) + Du(X_s)b(X_s)\big) ds$$

and using that u solves (19), i.e., $\boxed{\mathcal{L}u + Du\, b = \lambda u - b}$, we can replace

$$\int_0^t \mathcal{L}u(X_s)ds + \int_0^t Du(X_s)b(X_s)ds$$

with $-\int_0^t b(X_s)ds + \lambda \int_0^t u(X_s)ds = x - X_t + L_t + \lambda \int_0^t u(X_s)ds$ (see the SDE) and obtain the identity, $\mathbb{P}$-a.s., $t \geqslant 0$,

$$u(X_t) - u(x)$$
$$= x - X_t + L_t + \lambda \int_0^t u(X_s) ds + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} [u(X_{s-} + x) - u(X_{s-})] \tilde{N}(ds, dx).$$

II Step. Let now X and Y be two solutions (starting at $x \in \mathbb{R}^d$). By the previous identity , $\mathbb{P}$-a.s.,

$$X_t - Y_t = [u(Y_t) - u(X_t)]$$
$$+ \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} [u(X_{s-} + x) - u(X_{s-}) - u(Y_{s-} + x) + u(Y_{s-})] \tilde{N}(ds, dx)$$
$$+ \lambda \int_0^t [u(X_s) - u(Y_s)] ds.$$

Now in order to apply the Gronwall lemma to $\mathbb{E}|X_t - Y_t|^2$ and get uniqueness we need

$$\boxed{\|Du_\lambda\|_0 < 1 \quad \text{and} \quad 2\gamma > \alpha_0.}$$

The condition $2\gamma > \alpha_0$ (which implies $1+\gamma > \alpha_0$) is needed since

$$\mathbb{E}\left|\int_0^t \int_{\{|x|\leqslant 1\}} [u(X_{s-}+x) - u(X_{s-}) - u(Y_{s-}+x) + u(Y_{s-})]\tilde{N}(ds,dx)\right|^2$$

$$= \mathbb{E}\int_0^t ds \int_{\{|x|\leqslant 1\}} |u(X_{s-}+x) - u(X_{s-}) - u(Y_{s-}+x) + u(Y_{s-})|^2 \nu(dx)$$

$$\leqslant c\|u\|_{1+\gamma}^2 \int_0^t E|X_s - Y_s|^2 ds \int_{\{|x|\leqslant 1\}} |x|^{2\gamma}\nu(dx)$$

and $\int_{\{|x|\leqslant 1\}} |x|^{2\gamma}\nu(dx) < \infty$ only if $2\gamma > \alpha_0$. I have also used

Lemma

Let $\gamma \in [0,1]$ and $f \in C_b^{1+\gamma}(\mathbb{R}^d)$. Then for any $u, v \in \mathbb{R}^d$, $x \in \mathbb{R}^d$, with $|x| \leqslant 1$, we have

$$|f(u+x) - f(u) - f(v+x) + f(v)| \leqslant c_\gamma \|f\|_{1+\gamma} |u-v|\,|x|^\gamma, \quad \textit{with } c_\gamma = 3^{1-\gamma}.$$

Some ideas about the proof: **strong existence**

Since $\|Du\|_0 < 1/3$, the Hadamard theorem implies that the mapping

$$\psi : \mathbb{R}^d \to \mathbb{R}^d, \quad \psi(x) = x + u(x), \;\; x \in \mathbb{R}^d,$$

is a C^1-diffeomorphism from $\mathbb{R}^d$ onto $\mathbb{R}^d$. Moreover, $D\psi^{-1}$ is bounded on $\mathbb{R}^d$ and of class C^γ.

We can introduce the following *auxiliary SDE* with *usual Lipschitz conditions*.

$$Y_t = y + \int_0^t \tilde{b}(Y_s)ds \tag{20}$$

$$\int_0^t \int_{\{|z|\leqslant r\}} g(Y_{s-}, z)\tilde{N}(ds, dz) + \int_0^t \int_{\{|z|>r\}} g(Y_{s-}, z)N(ds, dz), \;\; t \geqslant 0,$$

where

$$\tilde{b}(y) = \lambda u(\psi^{-1}(y)) - \int_{\{|z|>r\}} [u(\psi^{-1}(y) \; + z) - u(\psi^{-1}(y))]\nu(dz) - \int_{\{r<|z|\leqslant 1\}} z\nu(dz)$$

$$= \lambda u(\psi^{-1}(y)) - \int_{\{|z|>r\}} [g(y, z) - z]\nu(dz) - \int_{\{r<|z|\leqslant 1\}} z\nu(dz)$$

and

$$g(y, z) = u(\psi^{-1}(y) \; + z) + z - u(\psi^{-1}(y)) =$$
$$= u(\psi^{-1}(y) \; + z) + z + \psi^{-1}(y) - \psi^{-1}(y) - u(\psi^{-1}(y))$$
$$= [\psi(\psi^{-1}(y) \; + z) - y], \quad y \in \mathbb{R}^d, \; z \in \mathbb{R}^d.$$

Strong existence

If (X_t) is a solution to (1) then by Ito's formula we easily get that

$$\big(\psi(X_t)\big)$$

is a solution to (20) starting from $y = \psi(x)$.

Viceversa if $x \in \mathbb{R}^d$ and we consider

$$Y_t^{\psi(x)} = Y_t,$$

$t \geqslant 0$, as the solution to the auxiliary SDE starting at $\psi(x)$ then we define

$$X_t = \psi^{-1}(Y_t).$$

A long computation shows that (X_t) is in fact a *strong solution* to the initial SDE starting at $x \in \mathbb{R}^d$.

On Schauder estimates

We first prove Schauder estimates for the resolvent equation with b constant, i.e.,

$$b(x) = b_0, \quad x \in \mathbb{R}^d.$$

We stress that the Schauder constant c in (21) is independent of b_0; this fact is needed to treat $\alpha = 1$ when b is variable.

Theorem (P. 2012)

Assume (H1), (H2). Let $\boxed{\alpha \in (0,2)}$ *and* $\beta \in (0,1)$ *be such that* $1 < \alpha + \beta < 2$. *Then, for any* $\lambda > 0$, $b_0 \in \mathbb{R}^d$, $g \in C_b^\beta(\mathbb{R}^d)$, *there exists a unique solution* $u = u_\lambda \in C_b^{\alpha+\beta}(\mathbb{R}^d)$ *to the equation*

$$\lambda u - \mathcal{L}u - b_0 \cdot Du = g$$

on $\mathbb{R}^d$. *In addition there exists a constant* c *independent of* g, u, b_0 *and* $\lambda > 0$ *such that*

$$\lambda \|u\|_0 + \lambda^{\frac{\alpha+\beta-1}{\alpha}} \|Du\|_0 + [Du]_{\alpha+\beta-1} \leqslant c\|g\|_\beta. \qquad (21)$$

Recall $[Du]_{\alpha+\beta-1} = \sup_{x \neq y} \frac{|Du(x)-Du(y)|}{|x-y|^{\alpha+\beta-1}}$.

In the proof we first introduce the α-stable Markov semigroup (P_t) acting on $C_b(\mathbb{R}^d)$ and associated to $\mathcal{L} + b_0 \cdot Du$, i.e.,

$$P_t f(x) = \int_{\mathbb{R}^d} f(z + tb_0)\, p_t(z - x) dz, \quad t > 0,\ f \in C_b(\mathbb{R}^d),\ x \in \mathbb{R}^d,$$

where p_t is the density of L_t, and $P_0 = I$.

Then we consider the bounded function $u = u_\lambda$,

$$u(x) = \int_0^\infty e^{-\lambda t} P_t g(x) dt, \quad x \in \mathbb{R}^d. \tag{22}$$

and show that u belongs to $C_b^{\alpha+\beta}(\mathbb{R}^d)$, verifies (21) and solves the equation. To this purpose we also need a maximum principle.

We consider now $b \in C_b^\beta(\mathbb{R}^d, \mathbb{R}^d)$.

Theorem (P. 2012)

Assume (H1) and (H2). Let $\alpha \geqslant 1$ *and* $\beta \in (0,1)$ *be such that* $1 < \alpha + \beta < 2$. *Then, for any* $\lambda > 0$, $g \in C_b^\beta(\mathbb{R}^d)$, *there exists a unique solution* $u = u_\lambda \in C_b^{\alpha+\beta}(\mathbb{R}^d)$ *to*

$$\lambda u - \mathcal{L}u - b \cdot Du = g$$

on $\mathbb{R}^d$. *Moreover, for any* $\omega > 0$, *there exists* $c = c(\omega)$, *independent of* g *and* u, *such that*

$$\lambda \|u\|_0 + [Du]_{\alpha+\beta-1} \leqslant c\|g\|_\beta, \quad \lambda \geqslant \omega. \tag{23}$$

Finally, we have $\lim_{\lambda\to\infty} \|Du_\lambda\|_0 = 0$.

Remark Note that $\gamma = \alpha + \beta - 1$. Hence the condition $2\gamma > \alpha$ in the Itô-Tanaka trick becomes

$$\beta > 1 - \alpha/2.$$

The localization procedure, $\alpha = 1$.

Let $u \in C_b^{1+\beta}(\mathbb{R}^d)$ be a solution.

Let $r > 0$ and $\xi \in C_0^\infty(\mathbb{R}^d)$ such that $\xi(x) = 1$ if $|x| \leqslant r$ and $\xi(x) = 0$ if $|x| > 2r$.

Let now $x_0 \in \mathbb{R}^d$ and define $\rho(x) = \xi(x - x_0)$, $x \in \mathbb{R}^d$, and $v = u\rho$. Then

$$\lambda v(x) - \mathcal{L}v(x) - b(x_0) \cdot Dv(x) = f_1(x) + f_2(x) + f_3(x) + f_4(x), \quad x \in \mathbb{R}^d,$$

where

$$f_1(x) = \rho(x)g(x), \qquad f_2(x) = (b(x) - b(x_0)) \cdot Dv(x),$$
$$f_3(x) = -u(x)[\mathcal{L}\rho(x) + b(x) \cdot D\rho(x)],$$
$$f_4(x) = -\int_{\mathbb{R}^d} (\rho(x+y) - \rho(x))(u(x+y) - u(x))\, \nu(dy), \quad x \in \mathbb{R}^d.$$

The point is that we know

$$[Dv]_\beta \leqslant C_1(\|f_1\|_\beta + \|f_2\|_\beta + \|f_3\|_\beta + \|f_4\|_\beta), \tag{24}$$

where the constant C_1 is independent of x_0 and λ and so we can continue.

A remark on the Zvonkin method

To simplify we still consider $\mathbb{R}$. We have

$$X_t = x + W_t + \int_0^t b(X_s)ds.$$

Now if $g(t,x)$ is a "regular" solution of

$$\begin{cases} \partial_t g + Lg = 0 \quad \text{on } [0,T] \times \mathbb{R}, \\ g(T,x) = x \end{cases}$$

then by Itô's formula:

$$X_T = g(0,x) + \int_0^T \partial_x g(s, X_s) dW_s$$