Well-posedness and numerical approximation of distributional solutions of non-local equations of porous medium type

Espen R. Jakobsen

Norwegian University of Science and Technology (NTNU)

www.math.ntnu.no/∼erj

Joint work with J. Endal (NTNU) and F. Del Teso (BCAM)

3rd Conference on Nonlocal Operators and PDEs

The main results

$$(\text{IVP}) \qquad \begin{cases} \partial_t u - \mathcal{L}[\varphi(u)] = 0 & \text{in} \quad \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x) & \text{in} \quad \mathbb{R}^N. \end{cases}$$

where

(i) $\varphi(\cdot)$ continuous, non-decreasing

(ii) $\mathcal{L}$ symmetric non-local degenerate elliptic (generator of Levy proc.)

The main results

$$
\text{(IVP)} \qquad \begin{cases} \partial_t u - \mathcal{L}[\varphi(u)] = 0 & \text{in} \quad \mathbb{R}^N \times (0, T), \\ u(x, 0) = u_0(x) & \text{in} \quad \mathbb{R}^N. \end{cases}
$$

where

(i) $\varphi(\cdot)$ continuous, non-decreasing

(ii) $\mathcal{L}$ symmetric non-local degenerate elliptic (generator of Levy proc.)

Main results:

(a) Uniqueness of bounded distributional sol'ns with $u - u_0 \in L^1$

(b) Existstence in $L^1 \cap L^\infty$

(c) Convergent numerical methods

Porous medium equations

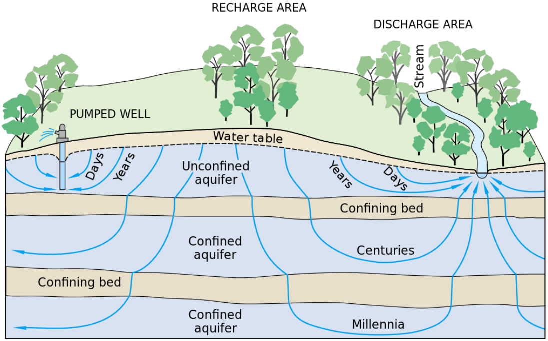

Figure: From U.S. Geological Survey Circular 1139.

- Engineering applications: gound water, oil and gas industri, ...
- Example: Forchheimer equations for height of water table h:

$$\partial_t h = (\partial_x^2 + \partial_y^2) h^2.$$

Local equations of porous medium type

- **Porous medium equations:** $\boxed{u_t = \Delta(u^m), \quad m \in (0,\infty)}$

  $m > 1$: Slow diffusion

  Non-smooth sol'ns, mass conservation, compact support

  $m = 1$: Heat equation

  Smooth sol'ns, mass conservation, str. positive

  $m < 1$: Fast diffusion

  Smooth sol'ns, conservation?? (extinction...), str. positive

Local equations of porous medium type

- **Porous medium equations:** $\boxed{u_t = \Delta(u^m), \quad m \in (0,\infty)}$

  $m > 1$: Slow diffusion

  Non-smooth sol'ns, mass conservation, compact support

  $m = 1$: Heat equation

  Smooth sol'ns, mass conservation, str. positive

  $m < 1$: Fast diffusion

  Smooth sol'ns, conservation?? (extinction...), str. positive

- **Stefan Problems:** $$\boxed{u_t = \Delta(u-1)^+ = \begin{cases} 0 & u \leq 1 \\ \Delta u & u > 1 \end{cases}}$$

  Phase transitions, free boundaries, strongly degenerate...

Local equations of porous medium type

- **Porous medium equations:** $u_t = \Delta(u^m), \quad m \in (0, \infty)$

  $m > 1$: Slow diffusion

  Non-smooth sol'ns, mass conservation, compact support

  $m = 1$: Heat equation

  Smooth sol'ns, mass conservation, str. positive

  $m < 1$: Fast diffusion

  Smooth sol'ns, conservation?? (extinction...), str. positive

- **Stefan Problems:** $$u_t = \Delta(u-1)^+ = \begin{cases} 0 & u \leq 1 \\ \Delta u & u > 1 \end{cases}$$

  Phase transitions, free boundaries, strongly degenerate...

- **Generalised PME:** $u_t = \Delta\varphi(u)$

  φ non-decreasing, continuous – contains all models.
  [Vazquez' book]

Non-local equations of porous medium type

$$(P1) \qquad \partial_t u = -(-\Delta)^s u^m, \qquad s \in (0,1), \quad m \in (0,\infty).$$

$$(P2) \qquad \partial_t u = \mathcal{L}[\varphi(u)], \qquad \varphi \text{ continuous non-decreasing}$$

Non-local equations of porous medium type

$$(P1) \qquad \partial_t u = -(-\Delta)^s u^m, \qquad s \in (0,1), \quad m \in (0,\infty).$$

$$(P2) \qquad \partial_t u = \mathcal{L}[\varphi(u)], \qquad \varphi \text{ continuous non-decreasing}$$

$(P1)$: Well-posedness, smooth, no compact support $m > 1$, extinction $m < 1$ ...

Vazquez, De Pablo, Quiros, Rodriguez, Bonforte, Stan, Del Teso, Brändle, Grillo, Muratori, Punzo, ...

Uniqueness typically needs finite energy or u_t in L^1_{loc}.

Non-local equations of porous medium type

$$(P1) \qquad \partial_t u = -(-\Delta)^s u^m, \qquad s \in (0,1), \quad m \in (0,\infty).$$

$$(P2) \qquad \partial_t u = \mathcal{L}[\varphi(u)], \qquad \varphi \text{ continuous non-decreasing}$$

$(P1)$: Well-posedness, smooth, no compact support $m > 1$, extinction $m < 1$ ...

Vazquez, De Pablo, Quiros, Rodriguez, Bonforte, Stan, Del Teso, Brändle, Grillo, Muratori, Punzo, ...

Uniqueness typically needs finite energy or u_t in L^1_{loc}.

$(P2)$: [Andreu, Mazon, Rossi, Toledo]: $\mathcal{L}$ nonlocal bounded operator...

[de Pablo et al. '16]: $\mathcal{L} \approx -(-\Delta)^s$ ... finite energy ...

Our results: General $\mathcal{L}$, weakest solution concept

Non-local equations of porous medium type

$$(P3) \qquad \partial_t u = \mathcal{L}[\varphi(u)] - \nabla \cdot f(u)$$

$$(P4) \qquad \partial_t u = \operatorname{div}\Big(u^k \nabla[(-\Delta)^{-s} u^m]\Big)$$

$(P3)$: Convection-diffusion, shock solutions possible.

ERJ, Alibaud, Cifani, Endal, Karlsen, Ulusoy,
Bieler, Karch, Woyczynski, ... , Andreianov, Droniou, Imbert,
... , Kiselev, ... , Dong, ... , Silvestre, ... , Jakubowski, ...

Existence and uniqueness under additional entropy conditions.

Non-local equations of porous medium type

$$(P3) \qquad \partial_t u = \mathcal{L}[\varphi(u)] - \nabla \cdot f(u)$$

$$(P4) \qquad \partial_t u = \mathrm{div}\Big(u^k \nabla[(-\Delta)^{-s} u^m]\Big)$$

$(P3)$: Convection-diffusion, shock solutions possible.

ERJ, Alibaud, Cifani, Endal, Karlsen, Ulusoy,
Bieler, Karch, Woyczynski, ... , Andreianov, Droniou, Imbert,
... , Kiselev, ... , Dong, ... , Silvestre, ... , Jakubowski, ...

Existence and uniqueness under additional entropy conditions.

$(P4)$: No uniqueness for dim > 1, solutions with compact supp. ...

Caffarelli, Vazquez, Bieler, Karch, Monneau, Imbert, Stan, Del Teso ...

Our non-local problem

$$\text{(IVP)} \qquad \begin{cases} \partial_t u - \mathcal{L}[\varphi(u)] = 0 & \text{in} \quad \mathbb{R}^N \times (0, T), \\ u(x,0) = u_0(x) & \text{in} \quad \mathbb{R}^N. \end{cases}$$

where

$$\mathcal{L}[\phi](x) = \int_{|z|>0} \Big(\phi(x+z) - \phi(x) - 1_{|z|<1}\, z \cdot D\phi(x)\Big)\, \mu(dz).$$

Assumptions:

(i) $\varphi(\cdot)$ is continuous and non-decreasing

(ii) $\mu \geq 0$, symmetric Radon measure, $\int_{|z|>0} |z|^2 \wedge 1\ \mu(dz) < \infty$.

(iii) $u_0 \in L^\infty(\mathbb{R}^N)$.

Remarks on our non-local problem

Conditions of φ:

- Includes porous medium, fast diffusion, and Stefan problem.
- Possibly strongly degenerate: $\varphi = const$ on interval.

Remarks on our non-local problem

Conditions of φ:

- Includes porous medium, fast diffusion, and Stefan problem.
- Possibly strongly degenerate: $\varphi = const$ on interval.

Why this $\mathcal{L}$??

- "Most general (symmetric, linear) non-local operator preserving the maximum principle"
- 1-1 correspondence to generators of (symmetric) pure-jump Levy processes.
- Includes relevant diffusion models from Finance.
- Includes natural numerical approximations of $\mathcal{L} + \sum \sigma_i \sigma_j \partial_i \partial_j$!

Remarks and examples on $\mathcal{L}$

- $\mathcal{L}$ can be degenerate (in some directions).

- Measure μ in $\mathcal{L}$ can be singular.

Remarks and examples on $\mathcal{L}$

- $\mathcal{L}$ can be degenerate (in some directions).
- Measure μ in $\mathcal{L}$ can be singular.
- Ex 1: $\mathcal{L} = -(-\Delta)^s$ for $s \in (0,1)$:

$$\mu(dz) = \frac{c_s}{|z|^{N+2s}} dz.$$

Remarks and examples on $\mathcal{L}$

- $\mathcal{L}$ can be degenerate (in some directions).
- Measure μ in $\mathcal{L}$ can be singular.
- Ex 1: $\mathcal{L} = -(-\Delta)^s$ for $s \in (0,1)$:

$$\mu(dz) = \frac{c_s}{|z|^{N+2s}} dz.$$

- Ex 2: $\mathcal{L}u(x) = \frac{1}{h^2}\Big(u(x+h) - 2u(x) + u(x-h)\Big)$:

$$\mu(dz) = \frac{1}{h^2}\Big(\delta_h(dz) + \delta_{-h}(dz)\Big).$$

Remarks and examples on $\mathcal{L}$

- $\mathcal{L}$ can be degenerate (in some directions).
- Measure μ in $\mathcal{L}$ can be singular.
- Ex 1: $\mathcal{L} = -(-\Delta)^s$ for $s \in (0,1)$:

$$\mu(dz) = \frac{c_s}{|z|^{N+2s}} dz.$$

- Ex 2: $\mathcal{L}u(x) = \frac{1}{h^2}\Big(u(x+h) - 2u(x) + u(x-h)\Big)$:

$$\mu(dz) = \frac{1}{h^2}\Big(\delta_h(dz) + \delta_{-h}(dz)\Big).$$

OBS: $\int |z|^2 \wedge 1 \; \mu(dz) \leq 2$ for all $h > 0$ ... equi-bounded ...

Our main result I: Well-posedness

$$\text{(IVP)} \qquad \begin{cases} \partial_t u - \mathcal{L}[\varphi(u)] = 0 & \text{in} \quad \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x) & \text{in} \quad \mathbb{R}^N. \end{cases}$$

Theorem (Uniqueness – [Endal-ERJ-del Teso], preprint 2015)

There is at most one distributional solution u of (IVP) such that $u \in L^\infty$ and $u - u_0 \in L^1$.

Theorem (Existence – [Endal-ERJ-del Teso], preprint 2015)

If $u_0 \in L^1 \cap L^\infty$, then there exists a (unique) distributional solution u of (IVP) and

$$u \in L^\infty(Q_T) \cap L^1(Q_T) \cap C([0,T]; L^1_{loc}(\mathbb{R}^N)).$$

Remarks

1. Weakest assumptions on φ

   ... same as [DePablo et al. '16]

Remarks

1. Weakest assumptions on φ

   ... same as [DePablo et al. '16]

2. Weakest assumptions on $\mathcal{L}$

   Before: $-(-\Delta)^{\frac{\alpha}{2}}$ or *comparable* operators

   Now: *ANY* generator of symmetric pure jump Levy process

   ... but no x-dependence as in [DePablo et al. '16]

Remarks

1. Weakest assumptions on φ

   ... same as [DePablo et al. '16]

2. Weakest assumptions on $\mathcal{L}$

   Before: $-(-\Delta)^{\frac{\alpha}{2}}$ or *comparable* operators

   Now: *ANY* generator of symmetric pure jump Levy process

   ... but no x-dependence as in [DePablo et al. '16]

3. Weakest solution concept

   No energy required ...

   ... but we need L^{∞} solutions

Remarks

1. Weakest assumptions on φ

   ... same as [DePablo et al. '16]

2. Weakest assumptions on $\mathcal{L}$

   Before: $-(-\Delta)^{\frac{\alpha}{2}}$ or *comparable* operators

   Now: *ANY* generator of symmetric pure jump Levy process

   ... but no x-dependence as in [DePablo et al. '16]

3. Weakest solution concept

   No energy required ...

   ... but we need L^∞ solutions

4. Proof: Uniqueness is hard! ... we will discuss it later ...

Numerical discretization - $\mathcal{L}$

z-grid parameter: $h > 0$

$$\mathcal{L}_h U(x) := \sum_{\alpha} \big[U(x+z_\alpha) - U(x)\big]\omega_\alpha, \qquad \omega_\alpha \geq 0.$$

Numerical discretization - $\mathcal{L}$

z-grid parameter: $h > 0$

$$\mathcal{L}_h U(x) := \sum_{\alpha} \big[U(x + z_\alpha) - U(x)\big]\omega_\alpha, \qquad \omega_\alpha \geq 0.$$

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 = \int_{|z|<h} \ldots + \int_{|z|\geq h} \ldots \approx \mathcal{L}_h = \mathcal{L}_{1,h} + \mathcal{L}_{2,h}$$

Numerical discretization - $\mathcal{L}$

z-grid parameter: $h > 0$

$$\mathcal{L}_h U(x) := \sum_{\alpha} \big[U(x + z_\alpha) - U(x)\big]\omega_\alpha, \qquad \omega_\alpha \geq 0.$$

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 = \int_{|z|<h} \ldots + \int_{|z|\geq h} \ldots \approx \mathcal{L}_h = \mathcal{L}_{1,h} + \mathcal{L}_{2,h}$$

1. Diffusion correction – small jumps:

$$\boxed{\mathcal{L}_{1,h} U(x) = \tfrac{1}{2}\tfrac{U(x+h)-2U(x)+U(x+h)}{h^2} \int_{|z|<h} |z|^2 \mu(dz)}$$

Numerical discretization - $\mathcal{L}$

z-grid parameter: $h > 0$

$$\mathcal{L}_h U(x) := \sum_\alpha \big[U(x+z_\alpha) - U(x)\big]\omega_\alpha, \qquad \omega_\alpha \geq 0.$$

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 = \int_{|z|<h} \dots + \int_{|z|\geq h} \dots \approx \mathcal{L}_h = \mathcal{L}_{1,h} + \mathcal{L}_{2,h}$$

1. Diffusion correction – small jumps:

$$\boxed{\mathcal{L}_{1,h} U(x) = \tfrac{1}{2}\tfrac{U(x+h)-2U(x)+U(x+h)}{h^2} \int_{|z|<h} |z|^2 \mu(dz)}$$

2. Lagrange interpolation quadrature:

$$\boxed{\mathcal{L}_{h,2} U = \mathcal{L}_2[iU]} \qquad U(x) \approx iU(x) = \textstyle\sum_\alpha U(z_\alpha) p_\alpha(x)$$

Numerical discretization - $\mathcal{L}$

z-grid parameter: $h > 0$

$$\mathcal{L}_h U(x) := \sum_\alpha \big[U(x+z_\alpha) - U(x)\big]\omega_\alpha, \qquad \omega_\alpha \geq 0.$$

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 = \int_{|z|<h} \dots + \int_{|z|\geq h} \dots \approx \mathcal{L}_h = \mathcal{L}_{1,h} + \mathcal{L}_{2,h}$$

1. Diffusion correction – small jumps:

$$\boxed{\mathcal{L}_{1,h} U(x) = \frac{1}{2}\frac{U(x+h)-2U(x)+U(x+h)}{h^2}\int_{|z|<h}|z|^2\mu(dz)}$$

2. Lagrange interpolation quadrature:

$$\boxed{\mathcal{L}_{h,2} U = \mathcal{L}_2[iU]} \qquad U(x) \approx iU(x) = \textstyle\sum_\alpha U(z_\alpha)p_\alpha(x)$$

3. Other approximations ... BUT NEED $\omega_\alpha \geq 0$! ($\Rightarrow$ positive)

Numerical discretization - time

Time step: $\Delta t > 0$

$$\frac{U^{n+1} - U^n}{\Delta t} = \alpha \mathcal{L}_{1,h}[\varphi(U^n)] + \beta \mathcal{L}_{2,h}[\varphi(U^{n+1})], \qquad \alpha, \beta \in [0,1].$$

1. The θ-method:

$$\boxed{\mathcal{L}_h := \mathcal{L}_{1,h} = \mathcal{L}_{2,h}, \quad \theta := \alpha = 1 - \beta}$$

$\theta = 1$ explicit, $\quad \theta = 0$ implicit, $\quad \theta = \frac{1}{2}$ Crank-Nicholson

Numerical discretization - time

Time step: $\Delta t > 0$

$$\frac{U^{n+1} - U^n}{\Delta t} = \alpha \mathcal{L}_{1,h}[\varphi(U^n)] + \beta \mathcal{L}_{2,h}[\varphi(U^{n+1})], \qquad \alpha, \beta \in [0,1].$$

1. The θ-method:

$$\boxed{\mathcal{L}_h := \mathcal{L}_{1,h} = \mathcal{L}_{2,h}, \quad \theta := \alpha = 1 - \beta}$$

$\theta = 1$ explicit, $\theta = 0$ implicit, $\theta = \frac{1}{2}$ Crank-Nicholson

2. Explicit-Implicit method:

$$\boxed{\mathcal{L}_{1,h} = \int_{|z|<1} \cdots, \quad \mathcal{L}_{2,h} = \int_{|z|\geq 1} \cdots, \quad \alpha = \beta = 1}$$

Numerical discretization - time

Time step: $\Delta t > 0$

$$\frac{U^{n+1} - U^n}{\Delta t} = \alpha \mathcal{L}_{1,h}[\varphi(U^n)] + \beta \mathcal{L}_{2,h}[\varphi(U^{n+1})], \qquad \alpha, \beta \in [0,1].$$

1. The θ-method:

$$\boxed{\mathcal{L}_h := \mathcal{L}_{1,h} = \mathcal{L}_{2,h}, \quad \theta := \alpha = 1 - \beta}$$

$\theta = 1$ explicit, $\quad \theta = 0$ implicit, $\quad \theta = \frac{1}{2}$ Crank-Nicholson

2. Explicit-Implicit method:

$$\boxed{\mathcal{L}_{1,h} = \int_{|z|<1} \cdots, \quad \mathcal{L}_{2,h} = \int_{|z|\geq 1} \cdots, \quad \alpha = \beta = 1}$$

3. Combinations... more operators $\mathcal{L}_{1,h}, \ldots, \mathcal{L}_{N,h}$ ...

Our main results II: Numerical approximations

Properties:

a. Monotone (CFL condition + $\omega_\alpha \geq 0$)

b. Stable in L^p, $p \in [1, \infty]$

c. Consistent (weakly in L^1 – assumptions!)

$\exists!$, max principle, L^1 contraction/compactness, mass conservation

Main result (Convergence – [Endal-ERJ-del Teso], in preparation)

Let U be the numerical solution, and u the solution of (IVP), then under general assumptions

$$U \underset{h,\Delta t \to 0}{\longrightarrow} u \qquad \text{in} \qquad C([0,T]; L^1_{\text{loc}}(\mathbb{R}^N))$$

Remarks

1. Proof: Compactness(!) + stability + uniqueness of limit

Remarks

1. Proof: Compactness(!) + stability + uniqueness of limit
2. Fully discrete method:

   Need x-grid $\quad \{x_\alpha\}_\alpha \supset x_{\alpha_1} + z_{\alpha_2} \quad$ for all α_1, α_2.

Remarks

1. Proof: Compactness(!) + stability + uniqueness of limit
2. Fully discrete method:

   Need x-grid $\{x_\alpha\}_\alpha \supset x_{\alpha_1} + z_{\alpha_2}$ for all α_1, α_2.
3. Extension – local equations:

   Finit difference approximations of $u_t = \Delta\varphi(u)$

Remarks

1. Proof: Compactness(!) + stability + uniqueness of limit
2. Fully discrete method:

   Need x-grid $\{x_\alpha\}_\alpha \supset x_{\alpha_1} + z_{\alpha_2}$ for all α_1, α_2.
3. Extension – local equations:

   Finit difference approximations of $u_t = \Delta\varphi(u)$
4. Fast diffusion case:

   First approximation and convergence result in nonlocal case

   New convergence result for local equations ???

Remarks

1. Proof: Compactness(!) + stability + uniqueness of limit
2. Fully discrete method:

   Need x-grid $\{x_\alpha\}_\alpha \supset x_{\alpha_1} + z_{\alpha_2}$ for all α_1, α_2.
3. Extension – local equations:

   Finit difference approximations of $u_t = \Delta\varphi(u)$
4. Fast diffusion case:

   First approximation and convergence result in nonlocal case

   New convergence result for local equations ???
5. Literature:

   Nonlocal-PM: Cifani-ERJ '14 (entropy sol'ns), Del Teso '14

   Local-PM: Eymard, Herbin, Galloet, Nochetto, Verdi, ...

   Other nonlocal: Many authors including myself...

The uniqueness proof

Uniqueness is hard to prove in this case because of

- the weak solution concept in combination with
- the generality of our assumptions.

The uniqueness proof

Uniqueness is hard to prove in this case because of

- the weak solution concept in combination with
- the generality of our assumptions.

Of several known potential ways to prove uniqueness...

- Oleinik's method (finite energy needed),
- L^1-contraction (u_t regular or entropy conditions),
- Duality methods,
- Resolvent based method of Brezis-Crandall,

... only the last one can work here!

The uniqueness proof of Brezis-Crandall (1979)

1 Assume two solution u, v.

Set $U = u - v$ and $\Phi = \varphi(u) - \varphi(v)$:

$$\partial_t U = \mathcal{L}\Phi \qquad \text{and} \qquad U(x,0) = 0.$$

2 Let $B_\varepsilon := (\varepsilon - \mathcal{L})^{-1}$ (resolvent), $\boxed{h_\varepsilon(t) = \int U\, B_\varepsilon U\, dx}$

3 $h_\varepsilon(t) = \int (\varepsilon - \mathcal{L})B_\varepsilon U\, B_\varepsilon U\, dx = \varepsilon \|B_\varepsilon U\|^2_{L^2} + \|(-\mathcal{L})^{1/2} B_\varepsilon U\|^2_{L^2}$

4 Show that $h_\varepsilon(t) \to 0$ as $\varepsilon \to 0$!

5 Conclusion: $U = (\varepsilon - \mathcal{L})B_\varepsilon U \to 0$ as $\varepsilon \to 0$ in $\mathcal{D}'$ (by 3, 4!) $\square$

On the uniqueness proof of Brezis-Crandall

- To do the proof you need:

  (i) Good properties of B_ε: e.g. it should be a bounded operator on each of the following spaces L^1, L^∞, C_b^∞.

  (ii) Need to show part 4: $h_\varepsilon \to 0$ as $\varepsilon \to 0$.

On the uniqueness proof of Brezis-Crandall

- To do the proof you need:

  (i) Good properties of B_ε: e.g. it should be a bounded operator on each of the following spaces L^1, L^∞, C_b^∞.

  (ii) Need to show part 4: $h_\varepsilon \to 0$ as $\varepsilon \to 0$.

- Brezis-Crandall: Local equation $u_t = \Delta\varphi(u)$. There:

  Δ har very nice properties: smoothing, local, ...

  B_ε has a nice and explicit kernal form which (i) and (ii) follows rather easily.

On the uniqueness proof of Brezis-Crandall

- To do the proof you need:

  (i) Good properties of B_ε: e.g. it should be a bounded operator on each of the following spaces L^1, L^∞, C_b^∞.

  (ii) Need to show part 4: $h_\varepsilon \to 0$ as $\varepsilon \to 0$.

- Brezis-Crandall: Local equation $u_t = \Delta\varphi(u)$. There:
  Δ har very nice properties: smoothing, local, ...

  B_ε has a nice and explicit kernal form which (i) and (ii) follows rather easily.

- For us: No smoothing, no expicit kernals, non-local...

  ... we need a diffent method

Uniqueness: Getting the properties of B_ε

- Properties of $B_\varepsilon = (\varepsilon - \mathcal{L})^{-1}$ obtained from study of

$$\varepsilon u - \mathcal{L} u = f.$$

Uniqueness: Getting the properties of B_ε

- Properties of $B_\varepsilon = (\varepsilon - \mathcal{L})^{-1}$ obtained from study of

$$\varepsilon u - \mathcal{L} u = f.$$

- Key (non-local) tool: Truncation $\mathcal{L} \to \mathcal{L}_r = \int_{|z|>r} \cdots$,

$$\varepsilon u - \mathcal{L}_r u = f.$$

Uniqueness: Getting the properties of B_ε

- Properties of $B_\varepsilon = (\varepsilon - \mathcal{L})^{-1}$ obtained from study of

$$\varepsilon u - \mathcal{L}u = f.$$

- Key (non-local) tool: Truncation $\mathcal{L} \to \mathcal{L}_r = \int_{|z|>r} \cdots$,

$$\varepsilon u - \mathcal{L}_r u = f.$$

- Prove results for this problem (uniformly in r)...
  Approximation, regualization, localization...
- Uniform in r a priori estimates, compactness, pass to the limit.

Uniqueness: Getting the properties of B_ε

- Properties of $B_\varepsilon = (\varepsilon - \mathcal{L})^{-1}$ obtained from study of

$$\varepsilon u - \mathcal{L}u = f.$$

- Key (non-local) tool: Truncation $\mathcal{L} \to \mathcal{L}_r = \int_{|z|>r} \cdots$,

$$\varepsilon u - \mathcal{L}_r u = f.$$

- Prove results for this problem (uniformly in r)...
  Approximation, regualization, localization...
- Uniform in r a priori estimates, compactness, pass to the limit.
- Need/use new Stroock-Varopolous inequalities: e.g.

$$\Big(Z(\psi), \mathcal{L}Z(\psi)\Big)_{L^2} = -\Big\|(-\mathcal{L})^{\frac{1}{2}}[Z(\psi)]\Big\|_{L^2}^2$$

with $(Z')^2 = \zeta'$, $Z(0) = 0$ ...

Uniqueness: Proving $h_\varepsilon \to 0$

- Prove that $\varepsilon B_\varepsilon[U] \to 0$ a.e. $\implies$ $h_\varepsilon \to 0$.

Uniqueness: Proving $h_\varepsilon \to 0$

- Prove that $\varepsilon B_\varepsilon[U] \to 0$ a.e. $\implies$ $h_\varepsilon \to 0$.
- By duality and compactness arguments, reduce to prove that

$$\varepsilon B_\varepsilon[\phi] \to 0 \qquad \text{for any} \qquad \phi \in C_c^\infty.$$

Uniqueness: Proving $h_\varepsilon \to 0$

- Prove that $\varepsilon B_\varepsilon[U] \to 0$ a.e. $\implies$ $h_\varepsilon \to 0$.
- By duality and compactness arguments, reduce to prove that

$$\varepsilon B_\varepsilon[\phi] \to 0 \qquad \text{for any} \qquad \phi \in C_c^\infty.$$

- Consider the equation for $v_\varepsilon = \varepsilon B_\varepsilon[\phi]$,

$$\varepsilon v - \mathcal{L} v = \varepsilon \phi.$$

Uniqueness: Proving $h_\varepsilon \to 0$

- Prove that $\varepsilon B_\varepsilon[U] \to 0$ a.e. $\implies$ $h_\varepsilon \to 0$.
- By duality and compactness arguments, reduce to prove that

$$\varepsilon B_\varepsilon[\phi] \to 0 \qquad \text{for any} \qquad \phi \in C_c^\infty.$$

- Consider the equation for $v_\varepsilon = \varepsilon B_\varepsilon[\phi]$,

$$\varepsilon v - \mathcal{L} v = \varepsilon \phi.$$

- A priori estimates $\|v_\varepsilon\|_{L^1} \leq \|\phi\|_{L^1} \ldots$ compactness, Barbalatt ...
$\implies$ $v_\varepsilon \to \tilde{v}$, $\tilde{v} \in C_0$, and $\mathcal{L}\tilde{v} = 0$.

Uniqueness: Proving $h_\varepsilon \to 0$

- Prove that $\varepsilon B_\varepsilon[U] \to 0$ a.e. $\implies$ $h_\varepsilon \to 0$.
- By duality and compactness arguments, reduce to prove that

$$\varepsilon B_\varepsilon[\phi] \to 0 \qquad \text{for any} \qquad \phi \in C_c^\infty.$$

- Consider the equation for $v_\varepsilon = \varepsilon B_\varepsilon[\phi]$,

$$\varepsilon v - \mathcal{L} v = \varepsilon \phi.$$

- A priori estimates $\|v_\varepsilon\|_{L^1} \leq \|\phi\|_{L^1}$... compactness, Barbalatt ...
$\implies$ $v_\varepsilon \to \tilde{v}$, $\tilde{v} \in C_0$, and $\mathcal{L}\tilde{v} = 0$.
- Conclude since $\tilde{v} = 0$ by new "Liouville like result":

  If $\mathrm{supp}\mu \neq \emptyset$ and $v \in C_0$ solves $\mathcal{L} v = 0$, then $v = 0$.

Uniqueness: Proving $h_\varepsilon \to 0$

- Prove that $\varepsilon B_\varepsilon[U] \to 0$ a.e. $\implies$ $h_\varepsilon \to 0$.
- By duality and compactness arguments, reduce to prove that

$$\varepsilon B_\varepsilon[\phi] \to 0 \qquad \text{for any} \qquad \phi \in C_c^\infty.$$

- Consider the equation for $v_\varepsilon = \varepsilon B_\varepsilon[\phi]$,

$$\varepsilon v - \mathcal{L} v = \varepsilon \phi.$$

- A priori estimates $\|v_\varepsilon\|_{L^1} \leq \|\phi\|_{L^1}$... compactness, Barbalatt ...
  $\implies$ $v_\varepsilon \to \tilde{v}$, $\tilde{v} \in C_0$, and $\mathcal{L}\tilde{v} = 0$.
- Conclude since $\tilde{v} = 0$ by new "Liouville like result":

  If $\mathrm{supp}\mu \neq \emptyset$ and $v \in C_0$ solves $\mathcal{L}v = 0$, then $v = 0$.
- OBS: No classical Liouville possible in this generality!!

References

Our results:

1. Endal, Jakobsen, and del Teso: *Uniqueness and properties of distributional solutions of nonlocal equations of porous medium type.* http://arxiv.org/abs/1507.04659

2. Endal, Jakobsen, and del Teso: *On numerical methods for distributional solutions of nonlocal equations of porous medium type.* In preparation.

References

Our results:

1. Endal, Jakobsen, and del Teso: *Uniqueness and properties of distributional solutions of nonlocal equations of porous medium type.* http://arxiv.org/abs/1507.04659

2. Endal, Jakobsen, and del Teso: *On numerical methods for distributional solutions of nonlocal equations of porous medium type.* In preparation.

Related results:

3. De Pablo, Quiros, Rodriguez, and Vazquez: *A general fractional porous medium equation.* Comm. Pure Appl. Math., 65(9), 2012.
4. De Pablo, Quiros, and Rodriguez: *Nonlocal filtration equation with rough kernels.* Nonlin. Analysis: Theory, Methods & Applications 137, 2016.
5. Brezis and Crandall: *Uniqueness of solutions of the initial-value problem for* $u_t - \Delta\varphi(u) = 0$. J. Math. Pures Appl., 58(2), 1979.
6. Cifani and Jakobsen. *Entropy solution theory for fractional degenerate convection-diffusion equations.* Ann. Inst. H. Poincare Anal. Non Lineaire 28(3), 2011.
7. Cifani and Jakobsen: *On numerical methods and error estimates for degenerate fractional convection-diffusion equations.* Numer. Math. 127(3), 2014.
8. Del Teso: *Finite difference method for a fractional porous medium equation.* Calcolo 51(4), 2014.