Improved Sobolev embeddings, profile decomposition, and global compactness for fractional Sobolev spaces

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3rd Conference on Nonlocal Operators and Partial Differential Equations

Bedlewo, 2016, June 27

Fractional Sobolev embeddings

Let $N \geq 1$ and for each $0 < s < N/2$ denote by $H^s(\mathbb{R}^N)$ the usual L^2-based fractional Sobolev spaces and $\dot{H}^s(\mathbb{R}^N)$ its homogeneous version defined via Fourier transform as the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|_{\dot{H}^s}^2 = \|(-\Delta)^{\frac{s}{2}} u\|_{L^2} = \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi \, .$$

We mainly focus our attention on fractional Sobolev embeddings $\dot{H}^s(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$,

$$(\star) \quad \|u\|_{L^{2^*}}^{2^*} \leq S^* \|u\|_{\dot{H}^s}^{2^*} \quad \forall u \in \dot{H}^s(\mathbb{R}^N),$$

where $2^* = 2N/(N-2s)$ is the critical Sobolev exponent.

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A naive approach to the validity of $(\star)$ is to study the variational problem

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[Cotsiolis-Tavoularis, *JMAA* 2004], following [Lieb, *Ann. Math.* 1983], proved

$$S^* \textit{is attained iff } u(x) = \frac{c}{(\lambda^2 + |x - x_0|^2)^{\frac{N-2s}{2}}} \quad \forall x \in \mathbb{R}^N,$$

where $c, \lambda \neq 0$ *are constants and* $x_0 \in \mathbb{R}^N$ *is a fixed point.*

See also [Chen-Li-Ou, *CPAM* 2006], and, for s in (0,1), [Frank-Seiringer, *JFA* 2008].

Note that the existence of a maximizer is not trivial since the embedding $(\star)$ is not compact, because of translation and dilation invariance. Indeed, if $u \in \dot{H}^s(\mathbb{R}^N)$ is an admissible function for S^*, the same holds for

$$u_{x_0,\lambda}(x) = \lambda^{-\frac{N-2s}{2}} u\left(\frac{x-x_0}{\lambda}\right),$$

for any $x_0 \in \mathbb{R}^N$ and any $\lambda > 0$.

In addition, $u_{x_0,\lambda}$ satisfies $\|u_{x_0,\lambda}\|_{L^{2^*}} = \|u\|_{L^{2^*}}$ and $\|u_{x_0,\lambda}\|_{\dot{H}^s} = \|u\|_{\dot{H}^s}$.

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Goals

- *Improving* the Sobolev inequality,

and

- investigating the lack of compactness in the Sobolev embeddings, trying to *restore* it.

1. Improving the Sobolev inequalities
2. Restoring compactness

Improved Sobolev embeddings

$\|u\|^{2^*}_{L^{2^*}} \leq S^* \|u\|^{2^*}_{\dot{H}^s}$ $(\star)$

By a refinement of the Sobolev embedding, one means that there exists a Banach function space X such that $\dot{H}^s \hookrightarrow X$ continuously (possibly $\dot{H}^s \hookrightarrow L^{2^*} \hookrightarrow X$) and, for some $0 < \theta < 1$ and some $C > 0$,

$$\|u\|_{L^{2^*}} \leq C \|u\|^{\theta}_{\dot{H}^s} \|u\|^{1-\theta}_{X}, \quad \forall u \in \dot{H}^s(\mathbb{R}^N)\,.$$

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$\Rightarrow$ $(\star)$

$\theta + (1-\theta) = 1$

(1) The simplest choice: the Lorentz space $X = L(2^*, \infty)$, (so that $L^{2^*} \hookrightarrow X$), where $\theta = 2/2^*$, proved by combining Peetre' Sobolev embedding [Peetre/Lions-Peetre, *C.R./Inst. Hautes Études* 1964] with Hölder's inequality. See, also, [Frank-Seiringer, *JFA* 2008].

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(2) In the same direction: the homogeneous Besov space of negative smoothness $X = \dot{B}^{-N/2^*}_{\infty,\infty}$, and still $\theta = 2/2^*$ and $L^{2^*} \hookrightarrow X$. See [Gérard-Meyer-Oru, *Sém. EDP Ecole Polytech.* 1997].

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$\Downarrow$ $\dot B^s_{2,\infty} \hookrightarrow L(2^*, \infty)$ [Chemin-Xu, *Ann. Sci. École Norm. Sup.* 1997]

Also: $X = \dot B^s_{2,\infty}$, an homogeneous Besov space of positive smoothness very close to $\dot H^s$ (so that one has $\dot H^s \hookrightarrow X \hookrightarrow L(2^*, \infty)$ but $X \not\hookrightarrow L^{2^*}$). See [Gérard, *ESAIM* 1998].

$\Uparrow$ $\dot B^s_{2,\infty} \hookrightarrow \dot B^{-N/2^*}_{\infty,\infty}$ (by Hölder's inequality)

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For any $1 \leq r < \infty$ and any $0 \leq \gamma \leq N$ let denote by $\mathcal{L}^{r,\gamma}$ the usual homogeneous Morrey space,

$$\|u\|^r_{\mathcal{L}^{r,\gamma}(\mathbb{R}^N)} := \sup_{R>0;\ x\in\mathbb{R}^n} R^{\gamma} \fint_{B_R(x)} |u|^r\, dy \,<\, \infty.$$

Notice that if $\gamma = N$ then the Morrey spaces $\mathcal{L}^{r,n}$ coincide with the usual Lebesgue spaces L^r for any $r \geq 1$; if $\gamma = 0$ then $\mathcal{L}^{r,0}$ coincide with L^∞.

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For $1 \leq r < 2^*$ we let $\gamma = r(N-2s)/2$ (so that $0 < \gamma < N$) and consider $X = \mathcal{L}^{r,r(N-2s)/2}$.

We have

Theorem 1 (Improved Sobolev inequality) [Palatucci-Pisante, *Calc. Var. PDE* 2014]

For any $0 < s < N/2$ there exists a constant C depending only on N and s such that, for any $2/2^* \leq \theta < 1$ and for any $1 \leq r < 2^*$,

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Remark. Note that, with this choice of the parameters, the norm of X has the same invariance property of the $\dot{H}^s$ norm. Also, **Theorem 1** implies the critical Sobolev inequality without assuming it. Indeed, by Hölder Inequality one can see that $L^{2^*} \hookrightarrow \mathcal{L}^{r,\frac{N-2s}{2}r} \equiv X$.

Proof of Theorem 1 - 1st version. We make use of a subtle estimate of the Riesz potentials on weighted Lebesgue spaces, as seen in [Sawyer-Wheeden, *Amer. J. Math.* 1992], using Calderón-Zygmund type techniques, much in the spirit of the fundamental Fefferman-Phong inequality. Then, we combine this estimate with a precise control on the $A_{p,q}$-constant associated to the weights in terms of the Morrey norm. □

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$$\mathcal{L}^{1,\alpha} \hookrightarrow \dot{B}^{-\alpha}_{\infty,\infty} \quad \text{for any } 0 < \alpha < N$$

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All in all, thanks to **Theorem 1** and **Lemma 1**, we now have the following chain of inclusions

$$\dot{B}^{s}_{2,\infty} \hookrightarrow L(2^*,\infty) \hookrightarrow \mathcal{L}^{r,r\frac{N-2s}{2}} \hookrightarrow \dot{B}^{-N/2^*}_{\infty,\infty},$$

and the refined Sobolev inequality does hold when X is any of these spaces.

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Disclaimer*: the result in **Lemma 1 is presumably well-known in the Navier-Stokes community (at least for nonhomogeneous spaces or for **α**=1, for which the space has the same scale-invariance of the equations. Since we were not able to find any precise references in the literature, we provided an elementary proof.

1. Improving the Sobolev inequalities

2. Restoring compactness

A first application of the improved Sobolev inequalities is the following lemma, which states that an appropriate scaling will assure a nontrivial weak-limit of any sequence in $\dot{H}^s$ uniformly bounded from below in the Lebesgue L^{2^*}-norm.

Lemma 2 (the fractional Translation Lemma) [Palatucci-Pisante, *Calc. Var. PDE* 2014]

Let $\{u_n\} \subset \dot{H}^s(\mathbb{R}^N)$ *a bounded sequence such that* $\inf_{n\in\mathbb{N}} \|u_n\|_{L^{2^*}} \geq c > 0$.
Then, up to subsequences, there exist a family of points $\{x_n\} \subset \mathbb{R}^N$ *and a family of positive numbers* $\{\lambda_n\} \subset (0,\infty)$ *such that*

$$\tilde{u}_n \rightharpoonup w \neq 0 \quad \textit{in } \dot{H}^s(\mathbb{R}^N),$$

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Proof.

The improved Sobolev inequality (**Theorem 1**) with $r = 2 \;\Rightarrow\; \inf \|u_n\|_{\mathcal{L}^{2,N-2s}} =: \beta > 0$.

Hence, $\exists \{x_n\}, \{\lambda_n\}$ s.t. $\lambda_n^{-2s} \int_{B_{\lambda_n}(x_n)} |u_n(y)|^2 dy \geq \tilde{\beta} > 0 \;\; \forall n$, just by the Morrey norms definition.

Passing to $\tilde{u}_n$ gives $\sup \|\tilde{u}_n\|_{\dot{H}^s} < \infty$, $\inf \int_{B_1} |\tilde{u}_n(x)|^2 dx \geq \tilde{\beta} > 0$.

Then (recall that $\dot{H}^s \hookrightarrow L^2_{\mathrm{loc}}$ is compact), up to subsequence, $\tilde{u}_n \rightharpoonup w$ in $\dot{H}^s(\mathbb{R}^N)$; hence, $w \neq 0$. □

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See also [Bellazzini-Frank-Visciglia, *Math. Ann.* 2014]. For s=1, see [Lieb, *Ann. Math.* 1983].

Combining **Lemma 2** together with the result in [Brezis-Lieb, *Proc. AMS* 1983] and the *convexity trick* by [P. L. Lions, *Rev. Iberoamer.* 1985], we can prove that the compactness of the optimizing sequences is restored when the natural invariance is taken into account.

Theorem 3 [Palatucci-Pisante, *Calc. Var. PDE* 2014]

Let $\{u_n\} \subset \dot{H}^s(\mathbb{R}^N)$ *be a maximizing sequence for the critical Sobolev inequality.*
Then, up to subsequences, there exist a sequence of points $\{x_n\} \subset \mathbb{R}^N$ *and a sequence of numbers* $\{\lambda_n\} \subset (0,\infty)$ *such that* $\tilde{u}_n(x) = \lambda_n^{(N-2s)/2} u_n\left(x_n + \lambda_n x\right)$ *converges to the maximizers* $u(x)$*, both in* L^{2^*} *and in* $\dot{H}^s$ *as* $n \to \infty$*.*

For the case when s=1, see [Lions, *Rev. Iberoamer.* 1985].

Theorem 4 (Profile decomposition) [Gérard, *ESAIM* 1998]

Let $\{u_n\}$ *be a bounded sequence in* $\dot{H}^s(\mathbb{R}^N)$. *Then, there exist a (at most countable) set* I, *a family of profiles* $\{\psi_j\} \subset \dot{H}^s(\mathbb{R}^N)$, *a family of points* $\{x_n^{(j)}\} \in \mathbb{R}^N$, *and a family of numbers* $\{\lambda_n^{(j)}\} \subset (0,\infty)$, *such that, up to subsequences, we have*

$$\left|\log\left(\frac{\lambda_n^{(i)}}{\lambda_n^{(j)}}\right)\right| + \left|\frac{(x_n^{(i)} - x_n^{(j)})}{\lambda_n^{(i)}}\right| \underset{n\to\infty}{\longrightarrow} \infty \ \text{ for } i \neq j,$$

$$u_n(x) = \sum_{j\in I} {\lambda_n^{(j)}}^{\frac{2s-N}{2}} \psi_j\left(\frac{x - x_n^{(j)}}{\lambda_n^{(j)}}\right) + r_n(x), \qquad \text{where } \lim_{n\to\infty} \|r_n\|_{L^{2^*}} = 0,$$

and

$$\|u_n\|^2_{\dot{H}^s} = \sum_{j\in I} \|\psi_j\|^2_{\dot{H}^s} + \|r_n\|^2_{\dot{H}^s} + o(1) \ \text{ as } n \to \infty.$$

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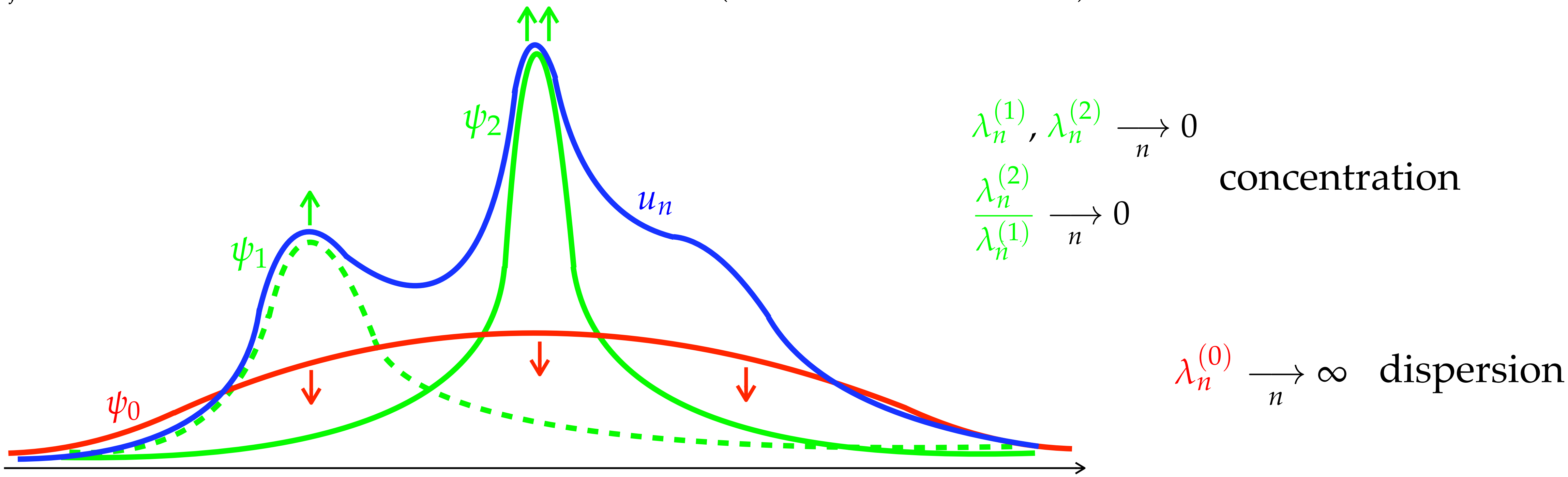
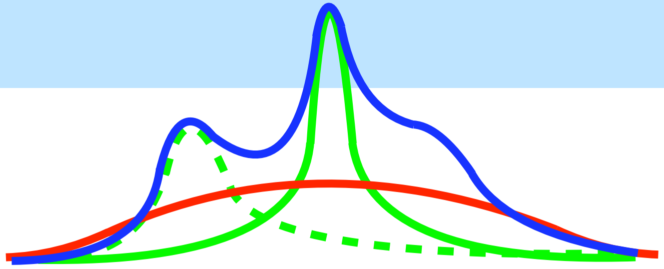
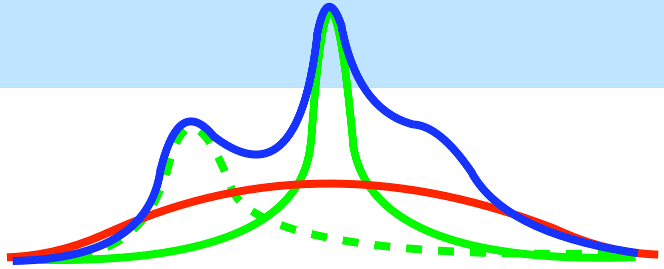
Remark. The sequence $\{u_n\}$ as $n \to \infty$ splits into the superposition of (differently modulating) profiles $\{\psi_j\}$ plus remainders $\{r_n\}$ negligible in L^{2^*} (but maybe not in $\dot{H}^s$).





Proof of Theorem 4 - 1st version (by Gérard, *ESAIM* 1998).



Proof of Theorem 4 - <u>1st version</u> (by Gérard, *ESAIM* 1998).

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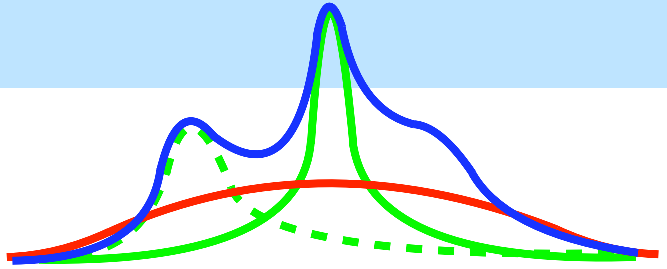

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We also need other tools to deal with cut-off functions and to provide a way to manipulate smooth truncations for the fractional Sobolev framework, as in the following

Lemma [Palatucci-Pisante, *Calc. Var. PDE* 2014]

Let $0 < s < N/2$ *and let* $u \in \dot{H}^s(\mathbb{R}^N)$. *Let* $\varphi \in C_0^\infty(\mathbb{R}^N)$ *and for each* $\lambda > 0$ *let* $\varphi_\lambda(x) := \varphi(\lambda^{-1}x)$. *Then*

$$u\varphi_\lambda \to 0 \text{ in } \dot{H}^s(\mathbb{R}^N) \text{ as } \lambda \to 0.$$

If, in addition, $\varphi \equiv 1$ *in a neighborhood of the origin, then* $u\varphi_\lambda \to u$ *in* $\dot{H}^s(\mathbb{R}^N)$ *as* $\lambda \to \infty$.

Lemma [Palatucci-Pisante, *Calc. Var. PDE* 2014]

Let $0 < s < N/2$, *let* $\Omega \subset \mathbb{R}^N$ *a bounded open set and let* $\varphi \in C_0^\infty(\mathbb{R}^N)$. *Then*

$$\varphi((-\Delta)^{\frac{s}{2}}u_n) - (-\Delta)^{\frac{s}{2}}(\varphi u_n) \to 0 \quad \text{in } L^2(\mathbb{R}^N)$$

whenever $u_n \rightharpoonup 0$ *in* $\dot{H}^s(\Omega)$ *as* $n \to \infty$;
i.e., the commutator $[\varphi, (-\Delta)^{\frac{s}{2}}] : \dot{H}^s(\Omega) \to L^2(\mathbb{R}^N)$ *is a compact operator.*

□

For any fixed $\lambda \in \mathbb{R}$, consider the following nonlinear problem

$$(\mathrm{P}_\lambda) \qquad (-\Delta)^s u - \lambda u - |u|^{2^*-2}u = 0 \ \text{ in } (\dot{H}^s(\Omega))',$$

i.e. the Euler-Lagrange equation $d\mathcal{E}(u) = 0$ corresponding to the differentiable functional

$$\mathcal{E}(u) = \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x - \frac{\lambda}{2}\int_\Omega |u|^2\,\mathrm{d}x - \frac{1}{2^*}\int_\Omega |u|^{2^*}\mathrm{d}x\,.$$

When $\lambda < \lambda_1$, although the functional possess the Mountain Pass geometry, the celebrated **Mountain Pass Lemma** does not apply because the **Palais-Smale (*PS*)** condition fails. More generally, the usual minimax scheme cannot be applied, and this is due to the presence of a limiting nonlinearity and it is related to the lack of compactness for the associated critical Sobolev embedding.

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For the case s=1, in [Brezis-Nirenberg, *CPAM* 1983] the authors circumvent this difficulty by proving that a local (*PS*)-condition holds for $\lambda < \lambda_1$ small enough. For $0 < s < 1$ see Barrios, Servadei, Soria, Valdinoci et Al.

Soon after, [Struwe, *Math. Z.* 1984], still in the local case $s = 1$, describes the precise mechanism responsible for the lack of the (*PS*)-condition; i.e., compactness for (*PS*)-sequences holds "*apart from jumps of the topological type of admissible functions*" <cit.>. The so-called "Global Compactness".

In order to state precisely this result, consider the limiting problem

$$(\mathrm{P}_0) \qquad (-\Delta)^s u - |u|^{2^*-2}u = 0 \ \text{ in } (\dot{H}^s(\Omega_0))'$$

where Ω_0 either the whole $\mathbb{R}^N$ or a half-space; i.e. the Euler-Lagrange equation $d\mathcal{E}^*(u) = 0$ corresponding to the energy functional $\mathcal{E}^* : \dot{H}^s(\Omega_0) \to \mathbb{R}$,

$$\mathcal{E}^*(u) = \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x - \frac{1}{2^*}\int_{\Omega_0} |u|^{2^*}\mathrm{d}x.$$

$$(\mathrm{P}_\lambda) \qquad (-\Delta)^s u - \lambda u - |u|^{2^*-2}u = 0 \text{ in } (\dot{H}^s(\Omega))'$$

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Theorem 5 (Fractional Global Compactness) [Palatucci-Pisante, *Nonlinear Anal.* 2015]

Let $\{u_n\}$ be a sequence in $\dot{H}^s(\Omega)$ such that $\mathcal{E}(u_n) \leq c$ and

$$d\mathcal{E}(u_n) \underset{n\to\infty}{\longrightarrow} 0 \text{ in } (\dot{H}^s(\Omega))'.$$

Then, there exists a (possibly trivial) solution $u^{(0)}$ to (P_λ) such that, for a renumbered subsequence of $\{u_n\}$, we have

$$u_n \underset{n\to\infty}{\rightharpoonup} u^{(0)} \text{ in } \dot{H}^s(\Omega).$$

Moreover,

either *the convergence is strong*

or *there exist a finite set $\mathcal{J} = \{1, 2, ..., J\}$, nontrivial solutions $\{u^{(j)}\}_{j\in\mathcal{J}}$ to (P_0) either in half spaces or in the entire space, $u^{(j)} \in \dot{H}^s(\Omega_0^{(j)})$, finitely many sequences of numbers $\{\lambda_n^{(j)}\}_{j\in\mathcal{J}} \subset (0,\infty)$ converging to zero, and finitely many sequences of points $\{x_n^{(j)}\}_{j\in\mathcal{J}} \subset \Omega$ such that, for a renumbered subsequence of $\{u_n\}$, we also have*

$$u_n^{(j)}(\cdot) := {\lambda_n^{(j)}}^{\frac{N-2s}{2}} u_n(x_n^{(j)} + \lambda_n^{(j)} \cdot) \underset{n\to\infty}{\rightharpoonup} u^{(j)}(\cdot) \text{ in } \dot{H}^s(\mathbb{R}^N).$$

In addition

$$u_n(\cdot) = u^{(0)}(\cdot) + \sum_{j=1}^{J} {\lambda_n^{(j)}}^{\frac{2s-N}{2}} u^{(j)}\left(\frac{\cdot - x_n^{(j)}}{\lambda_n^{(j)}}\right) + o(1) \text{ in } \dot{H}^s(\mathbb{R}^N),$$

and

$$\|u_n\|^2_{\dot{H}^s} = \sum_{j=0}^{J} \|u^{(j)}\|^2_{\dot{H}^s} + o(1) \text{ as } n \to \infty,$$

$$\mathcal{E}(u_n) = \mathcal{E}(u^{(0)}) + \sum_{j=1}^{J} \mathcal{E}^*(u^{(j)}) + o(1) \text{ as } n \to \infty.$$

For the case when s=1, see [Struwe, *Rev. Iberoamer.* 1984]; for s=2, see [Hebey-Robert, *Calc. Var. PDE* 2001] and [Grunau-Gazzola-Squassina, *Calc. Var. PDE* 2003].

Restoring compactness #3: Struwe's Global Compactness

(P$_\lambda$) $\quad (-\Delta)^s u - \lambda u - |u|^{2^*-2}u = 0 \text{ in } (\dot{H}^s(\Omega))'$

(P$_0$) $\quad (-\Delta)^s u - |u|^{2^*-2}u = 0 \text{ in } (\dot{H}^s(\Omega_0))'$

For s=1, the *Proof of Theorem 5* (by Struwe) consists in a subtle analysis concerning how the Palais-Smale condition fails for the energy functional, based on rescaling arguments, used in an iterated way to extract convergent subsequences with nontrivial limit, together with some slicing and extension procedures on the sequence of approximate solutions.

The same (not for free) for the case s=2.

(P$_\lambda$) $\quad (-\Delta)^s u - \lambda u - |u|^{2^*-2}u = 0$ in $(\dot{H}^s(\Omega))'$

(P$_0$) $\quad (-\Delta)^s u - |u|^{2^*-2}u = 0$ in $(\dot{H}^s(\Omega_0))'$

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The same (not for free) for the case s=2.

Gérard's Profile Decomposition

Let $\{u_n\}$ be a bounded sequence in $\dot{H}^s(\mathbb{R}^N)$.
Then, there exist a family of profiles $\{\psi_j\} \subset \dot{H}^s(\mathbb{R}^N)$, a family of points $\{x_n^{(j)}\} \in \mathbb{R}^N$, and a family of numbers $\{\lambda_n^{(j)}\} \subset (0,\infty)$, such that

$$\left|\log\left(\frac{\lambda_n^{(i)}}{\lambda_n^{(j)}}\right)\right| + \left|\frac{(x_n^{(i)} - x_n^{(j)})}{\lambda_n^{(i)}}\right| \underset{n\to\infty}{\longrightarrow} \infty$$

$$u_n(x) = \sum_{j\in I} {\lambda_n^{(j)}}^{\frac{2s-N}{2}} \psi_j\left(\frac{x - x_n^{(j)}}{\lambda_n^{(j)}}\right) + r_n(x),$$

where $\lim_{n\to\infty} \|r_n\|_{L^{2^*}} = 0,$

and $\|u_n\|^2_{\dot{H}^s} = \sum_{j\in I} \|\psi_j\|^2_{\dot{H}^s} + \|r_n\|^2_{\dot{H}^s} + o(1)$ *as* $n \to \infty$.

Struwe's Global Compactness

Let $\{u_n\}$ be a sequence in $\dot{H}^s(\Omega)$ such that $\mathcal{E}(u_n) \leq c$ and

$$d\mathcal{E}(u_n) \underset{n\to\infty}{\longrightarrow} 0 \text{ in } (\dot{H}^s(\Omega))'.$$

Then, there exists a (possibly trivial) solution $u^{(0)}$ to (P$_\lambda$) such that, for a renumbered subsequence of $\{u_n\}$, we have

$$u_n \underset{n\to\infty}{\rightharpoonup} u^{(0)} \text{ in } \dot{H}^s(\Omega).$$

Moreover,

either *the convergence is strong*

or *there exist nontrivial solutions $\{u^{(j)}\}_{j\in\mathcal{J}}$ to (P$_0$) either in half spaces or in the entire space,finitely many sequences of numbers $\{\lambda_n^{(j)}\}_{j\in\mathcal{J}} \subset (0,\infty)$ converging to zero, and finitely many sequences of points $\{x_n^{(j)}\}_{j\in\mathcal{J}} \subset \Omega$ such that,*

$$u_n(\cdot) = u^{(0)}(\cdot) + \sum_{j=1}^{J} {\lambda_n^{(j)}}^{\frac{2s-N}{2}} u^{(j)}\left(\frac{\cdot - x_n^{(j)}}{\lambda_n^{(j)}}\right) + o(1) \quad \text{in } \dot{H}^s(\mathbb{R}^N),$$

and

$$\|u_n\|^2_{\dot{H}^s} = \sum_{j=0}^{J} \|u^{(j)}\|^2_{\dot{H}^s} + o(1) \text{ as } n \to \infty,$$

$$\mathcal{E}(u_n) = \mathcal{E}(u^{(0)}) + \sum_{j=1}^{J} \mathcal{E}^*(u^{(j)}) + o(1) \text{ as } n \to \infty.$$

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$$u_n(\cdot) = u^{(0)}(\cdot) + \sum_{j=1}^{J} {\lambda_n^{(j)}}^{\frac{2s-N}{2}} u^{(j)}\left(\frac{\cdot - x_n^{(j)}}{\lambda_n^{(j)}}\right) + o(1) \quad \text{in } \dot{H}^s(\mathbb{R}^N),$$

and

$$\|u_n\|^2_{\dot{H}^s} = \sum_{j=0}^{J} \|u^{(j)}\|^2_{\dot{H}^s} + o(1) \text{ as } n \to \infty,$$

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$$u_n(x) = \sum_{j\in I} {\lambda_n^{(j)}}^{\frac{2s-N}{2}} \psi_j\left(\frac{x - x_n^{(j)}}{\lambda_n^{(j)}}\right) + r_n(x),$$

$$\text{where } \lim_{n\to\infty} \|r_n\|_{L^{2^*}} = 0,$$

$$\text{and } \|u_n\|^2_{\dot{H}^s} = \sum_{j\in I} \|\psi_j\|^2_{\dot{H}^s} + \|r_n\|^2_{\dot{H}^s} + o(1) \quad \text{as } n \to \infty.$$

Proof of Theorem 5.

Step 1. The sequence $\{u_n\}$ is bounded in $\dot{H}^s(\Omega)$. Straightforward from being a Palais-Smale sequence.

Step 2. The weak limit (up to subsequences) $u^{(0)}$ solves (P_λ). From *Step 1* and the weak continuity of $\mathrm{d}\mathcal{E}$.

If the convergence is strong we are done.

Step 3. If the convergence is not strong then $\{u_n\}$ contains further profiles.
Following from **Theorem 4** (Gérard's Profile Decomposition), and the structure of $\mathrm{d}\mathcal{E}$.

Step 4. The profiles $u^{(j)}$ solve (P_0) either in a half space or in the entire space and $\{x_n\} \subset \Omega$.
It follows by the assumption $d\mathcal{E}(u_n) \underset{n\to\infty}{\longrightarrow} 0$ in $\left(\dot{H}^s(\Omega)\right)'$, together with the invariance of the $\dot{H}^s$- and L^{2^*}-norm with respect to the right scaling.

Step 5. The profiles $u^{(j)}$ are in finite number.
It is a consequence of **Theorem 4**. It will suffice to show that the $\dot{H}^s$-energy of the nontrivial profiles $u^{(j)}$ is uniformly bounded from below, and this follows from the Sobolev inequality.

Step 6. The sequence of remainders $\{r_n\}$ converges strongly to 0 in $\dot{H}^s(\mathbb{R}^N)$.
It follows from the asymptotic orthogonality of the scaled profiles and the invariance of the $\dot{H}^s$-norm, together with the result in *Step 2* and *Step 4*, by iteratively applying the Brezis-Lieb lemma.

□

Improved Sobolev embeddings, profile decomposition, and global compactness for fractional Sobolev spaces

Giampiero Palatucci

thank you

Bedlewo, 2016, June 27th