From local to nonlocal potential estimates

Giuseppe Mingione

Nonlocal Bedlewo 2016

Part 1: Local Nonlinear Potential Theory

The classical potential estimates

- **Consider the model case**

$$-\triangle u = \mu \qquad \text{in } \mathbb{R}^n$$

The classical potential estimates

- **Consider the model case**

$$-\triangle u = \mu \qquad \text{in } \mathbb{R}^n$$

- **We have**

$$u(x) = \int G(x,y)\mu(y)$$

The classical potential estimates

- **Consider the model case**

$$-\triangle u = \mu \qquad \text{in } \mathbb{R}^n$$

- **We have**

$$u(x) = \int G(x,y)\mu(y)$$

where

$$G(x,y) \approx \begin{cases} |x-y|^{2-n} & \text{se } n > 2 \\ -\log|x-y| & \text{se } n = 2 \end{cases}$$

Estimates via Riesz potentials

- Previous formula gives

$$|u(x)| \lesssim \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x-y|^{n-2}} = I_2(|\mu|)(x)$$

Estimates via Riesz potentials

- Previous formula gives

$$|u(x)| \lesssim \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x-y|^{n-2}} = I_2(|\mu|)(x)$$

- while, after differentiation, we obtain

$$|Du(x)| \lesssim \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x-y|^{n-1}} = I_1(|\mu|)(x)$$

- **In bounded domains one uses**

$$\mathbf{I}^{\mu}_{\beta}(x,R) := \int_0^R \frac{|\mu|(B_\varrho(x))}{\varrho^{n-\beta}}\,\frac{d\varrho}{\varrho} \qquad \beta \in (0,n]$$

since

$$\begin{aligned}\mathbf{I}^{\mu}_{\beta}(x,R) &\lesssim \int_{B_R(x)} \frac{d|\mu|(y)}{|x-y|^{n-\beta}} \\ &= I_\beta(|\mu|\llcorner B_R(x))(x) \\ &\leq I_\beta(|\mu|)(x)\end{aligned}$$

for non-negative measures

What happens in the nonlinear case?

- **For instance for nonlinear equations with linear growth**

$$-\text{div } a(Du) = \mu$$

that is equations well posed in $W^{1,2}$ (p-growth and $p = 2$) that is

$$|\partial a(z)| \leq L \qquad \nu|\lambda|^2 \leq \langle \partial a(z)\lambda, \lambda\rangle$$

- **And degenerate ones like**

$$-\text{div } (|Du|^{p-2}Du) = \mu$$

- To be short, we shall concentrate on the case $p \geq 2$

Nonlinear potentials

- **The nonlinear Wolff potential is defined by**

$$\mathbf{W}^{\mu}_{\beta,p}(x,R) := \int_0^R \left(\frac{|\mu|(B_\varrho(x))}{\varrho^{n-\beta p}}\right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} \qquad \beta \in (0, n/p]$$

which for $p = 2$ reduces to the usual Riesz potential

$$\mathbf{I}^{\mu}_{\beta}(x,R) := \int_0^R \frac{\mu(B_\varrho(x))}{\varrho^{n-\beta}} \frac{d\varrho}{\varrho} \qquad \beta \in (0, n]$$

- **The nonlinear Wolff potential** plays in nonlinear potential theory the same role the Riesz potential plays in the linear one

The first nonlinear potential estimate

Theorem (Kilpeläinen & Malý, Acta Math. 94)

If u solves

$$-\text{div}\,(|Du|^{p-2}Du) = \mu$$

then

$$|u(x)| \lesssim \mathbf{W}^{\mu}_{1,p}(x,R) + \left(\fint_{B_R(x)} |u|^{p-1}\,dy \right)^{1/(p-1)}$$

holds

The first nonlinear potential estimate

Theorem (Kilpeläinen & Malý, Acta Math. 94)

If u solves

$$-\text{div}\,(|Du|^{p-2}Du) = \mu$$

then

$$|u(x)| \lesssim \mathbf{W}^{\mu}_{1,p}(x,R) + \left(\fint_{B_R(x)} |u|^{p-1}\,dy\right)^{1/(p-1)}$$

holds

where

$$\mathbf{W}^{\mu}_{1,p}(x,R) := \int_0^R \left(\frac{|\mu|(B_\varrho(x))}{\varrho^{n-p}}\right)^{1/(p-1)} \frac{d\varrho}{\varrho}$$

The first nonlinear potential estimate

Theorem (Kilpeläinen & Malý, Acta Math. 94)

If u solves

$$-\text{div}\,(|Du|^{p-2}Du) = \mu$$

then

$$|u(x)| \lesssim \mathbf{W}^{\mu}_{1,p}(x,R) + \left(\fint_{B_R(x)} |u|^{p-1}\,dy \right)^{1/(p-1)}$$

holds

where

$$\mathbf{W}^{\mu}_{1,p}(x,R) := \int_0^R \left(\frac{|\mu|(B_\varrho(x))}{\varrho^{n-p}} \right)^{1/(p-1)} \frac{d\varrho}{\varrho}$$

For $p = 2$ we are back to the Riesz potential $\mathbf{W}^{\mu}_{1,p} = \mathbf{I}^{\mu}_{2}$ - the above estimate is non-trivial already in this situation

Controlling the Wolff potential

$$\int_0^\infty \left(\frac{|\mu|(B_\varrho(x))}{\varrho^{n-\beta p}}\right)^{1/(p-1)} \frac{d\varrho}{\varrho} \lesssim I_\beta \left\{[I_\beta(|\mu|)]^{1/(p-1)}\right\}(x)$$

The quantity in the right-hand side is usually called Havin-Mazya potential

Approaching the gradient

The Wolff potential estimate for solutions still holds for solutions to

$$-\operatorname{div} a(x, Du) = \mu$$

with **measurable** coefficients

As for the gradient, the only thing that can be said is Gehring's lemma, i.e.,

$$Du \in L^{p+\delta}$$

for some small δ

A first gradient potential estimate

Theorem (Min., JEMS 11)

When $p = 2$, *if* u *solves*

$$-\operatorname{div} a(Du) = \mu$$

then

$$|Du(x)| \lesssim \mathbf{I}_1^{|\mu|}(x, R) + \fint_{B_R(x)} |Du| \, dy$$

holds

A first gradient potential estimate

Theorem (Min., JEMS 11)

When $p = 2$, *if* u *solves*

$$-\operatorname{div} a(Du) = \mu$$

then

$$|Du(x)| \lesssim \mathbf{I}_1^{|\mu|}(x,R) + \fint_{B_R(x)} |Du|\, dy$$

holds

For solutions in $W^{1,1}(\mathbb{R}^N)$ we have

$$|Du(x)| \lesssim \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x-y|^{n-1}} = I_1(|\mu|)(x)$$

The $p \neq 2$ case: a long path towards optimality

Theorem (Duzaar & Min., AJM 11)

When $p \geq 2$, if u solves

$$-\text{div}\, a(Du) = \mu$$

then

$$|Du(x)| \lesssim \mathbf{W}^{\mu}_{1/p,p}(x, R) + \fint_{B_R(x)} |Du|\, dy$$

holds

The $p \neq 2$ case: a long path towards optimality

Theorem (Duzaar & Min., AJM 11)

When $p \geq 2$, if u solves

$$-\text{div}\, a(Du) = \mu$$

then

$$|Du(x)| \lesssim \mathbf{W}^{\mu}_{1/p,p}(x,R) + \fint_{B_R(x)} |Du|\, dy$$

holds

where

$$\mathbf{W}^{\mu}_{1/p,p}(x,R) = \int_0^R \left(\frac{|\mu|(B_\varrho(x))}{\varrho^{n-1}} \right)^{1/(p-1)} \frac{d\varrho}{\varrho}$$

New viewpoint - Let's twist!!!

- Consider

$$-\text{div}\, v = \mu$$

with

$$v = |Du|^{p-2}Du$$

Indeed

Theorem (Kuusi & Min., CRAS 11 + ARMA 13)

If u solves

$$-\mathrm{div}\,(|Du|^{p-2}Du) = \mu$$

then

$$|Du(x)|^{p-1} \lesssim \mathbf{I}_1^{|\mu|}(x,R) + \left(\fint_{B_R(x)} |Du|\,dy\right)^{p-1}$$

holds

Indeed

Theorem (Kuusi & Min., CRAS 11 + ARMA 13)

If u solves

$$-\mathrm{div}\,(|Du|^{p-2}Du) = \mu$$

then

$$|Du(x)|^{p-1} \lesssim \mathbf{I}_1^{|\mu|}(x,R) + \left(\fint_{B_R(x)} |Du|\,dy\right)^{p-1}$$

holds

The theorem still holds for general equations of the type $-\mathrm{div}\,a(Du) = \mu$

A global estimate

Theorem (Kuusi & Min., CRAS 11 + ARMA 13)

If u solves

$$-\mathrm{div}\,(|Du|^{p-2}Du) = \mu$$

and decays naturally, then

$$|Du(x)|^{p-1} \lesssim \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x-y|^{n-1}} = I_1(|\mu|)(x)$$

Part 2: Nonlocal Nonlinear Potential Theory

The classical fractional Laplacean

$$(-\triangle)^{\alpha} u = f \qquad \text{for} \qquad 0 < s < 1$$

means that

$$\langle (-\triangle)^{\alpha} u, \varphi \rangle := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{[u(x)-u(y)][\varphi(x) - \varphi(y)]}{|x-y|^{n+2\alpha}} \, dx \, dy = \int_{\mathbb{R}^n} f \varphi \, dx$$

for every $\varphi \in C_0^{\infty}(\mathbb{R}^n)$

Nonlocal operators with measurable coefficients

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}[u(x)-u(y)][\varphi(x)-\varphi(y)]K(x,y)\,dx\,dy=\int_{\mathbb{R}^n} f\varphi\,dx$$

where

$$\frac{1}{\Lambda|x-y|^{n+2\alpha}}\leq K(x,y)\leq\frac{\Lambda}{|x-y|^{n+2\alpha}}\qquad\forall\ x,y\in\mathbb{R}^n,\ x\neq y$$

These correspond to linear elliptic equations of the type

$$-\mathrm{div}\,(A(x)Du)=f$$

where $A(x)$ is an elliptic matrix with measurable coefficients

Nonlinear nonlocal equations

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Phi(u(x)-u(y))[\varphi(x)-\varphi(y)]K(x,y)\,dx\,dy = \int_{\mathbb{R}^n} f\varphi\,dx$$

where

$$|\Phi(t)| \leq \Lambda|t|\,, \qquad \Phi(t)t \geq t^2, \qquad \forall\ t \in \mathbb{R}$$

These correspond to linear elliptic equations of the type

$$-\text{div}\, a(x, Du) = f$$

where $z \mapsto a(x,z)$ is strictly monotone with quadratic growth

The nonlocal p-Laplacean operator

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Phi(u(x)-u(y))[\varphi(x)-\varphi(y)]K(x,y)\,dx\,dy = \int_{\mathbb{R}^n} f\varphi\,dx$$

where this time

$$\frac{1}{\Lambda|x-y|^{n+p\alpha}} \leq K(x,y) \leq \frac{\Lambda}{|x-y|^{n+p\alpha}}$$

and

$$\Lambda^{-1}|t|^p \leq \Phi(t)t \leq \Lambda|t|^p$$

We consider the fractional p-Laplacean

$$\begin{aligned}
&\langle -\mathcal{L}_p u, \varphi \rangle \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(x) - u(y)|^{p-2}[u(x) - u(y)][\varphi(x) - \varphi(y)]K(x,y)\,dx\,dy \\
&\quad = \int_{\mathbb{R}^n} f\varphi\,dx
\end{aligned}$$

with

$$\frac{1}{\Lambda|x-y|^{n+p\alpha}} \leq K(x,y) \leq \frac{\Lambda}{|x-y|^{n+p\alpha}}$$

and

$$p \geq 2$$

for simplicity

This arises when minimizing fractional energies of the type

$$v \mapsto \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(x) - u(y)|^p K(x,y)\, dx\, dy$$

Nonlocal problems

We consider the nonlocal Dirichlet problem

$$\begin{cases} -\mathcal{L}_p u = 0 \text{ in } \Omega \\ \qquad u = g \text{ on } \mathbb{R}^n \setminus \Omega \end{cases}$$

where

$$g \in W^{\alpha,p}(\mathbb{R}^n)$$

The Tail

$$\mathrm{Tail}(v; x_0, r) := \left[r^{p\alpha} \int_{\mathbb{R}^n \setminus B_r(x_0)} \frac{|v(x)|^{p-1}}{|x-x_0|^{n+p\alpha}}\, dx \right]^{1/(p-1)}$$

Observe that $W^{\alpha,p}(\mathbb{R}^n)$-functions have finite tail. We can consider the tail space

$$L^{p-1}_{p\alpha}(\mathbb{R}^n) := \Big\{ v \in L^{p-1}_{\mathrm{loc}}(\mathbb{R}^n) \ :$$

$$\mathrm{Tail}(v; z, r) < \infty \ \ \forall\, z \in \mathbb{R}^n, \forall\, r \in (0, \infty) \Big\}$$

and assume that

$$g \in W^{\alpha,p}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^{p-1}_{p\alpha}(\mathbb{R}^n)$$

The sup-bound for the nonlocal p-Laplacean

Theorem (Di Castro & Kuusi & Palatucci, Ann. IHP, to appear)

Let $v \in W^{\alpha,p}(\mathbb{R}^n)$ be a weak solution. Let $B_r(x_0) \subset \Omega$; then the following estimate holds:

$$\sup_{B_{r/2}(x_0)} |v| \leq c \left(\fint_{B_r(x_0)} |v|^p \, dx \right)^{1/p} + c\,\mathrm{Tail}(v; x_0, r/2)$$

SOLA (detailed definition in the nonlocal setting)

Solutions obtained via limiting approximations

$$\begin{cases} -\mathcal{L}_p u_j = \mu_j \text{ in } \ \Omega \\ \qquad u_j = g_j \text{ on } \ \mathbb{R}^n \setminus \Omega\,, \end{cases}$$

where u_j converges to u a.e. in $\mathbb{R}^n$ and locally in $L^q(\mathbb{R}^n)$.
The sequence $\{\mu_j\} \subset C_0^\infty(\mathbb{R}^n)$ converges to μ weakly in the sense of measures in Ω and moreover satisfies

$$\limsup_{j\to\infty} |\mu_j|(B) \le |\mu|(\overline{B})$$

whenever B is a ball.
The sequence $\{g_j\} \subset C_0^\infty(\mathbb{R}^n)$ converges to g in the following sense: For all balls $B_r \equiv B_r(z)$ with center in z and radius $r > 0$, it holds that

$$g_j \to g \quad \text{in} \ \ W^{\alpha,p}(B_r)\,, \qquad \text{and} \qquad \lim_j \operatorname{Tail}(g_j - g; z, r) = 0$$

Wolff potentials bounds

Theorem (Kuusi & Min. & Sire, CMP 15)

Let $\mu \in \mathcal{M}(\mathbb{R}^n)$*,* $g \in W^{\alpha,p}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^{p-1}_{p\alpha}(\mathbb{R}^n)$*. Let* u *be a SOLA and assume that for a ball* $B_r(x_0) \subset \Omega$ *the Wolff potential* $\mathbf{W}^{\mu}_{\alpha,p}(x_0, r)$ *is finite.*

Then x_0 *is a Lebesgue point of* u *in the sense that there exists the precise representative of* u *at* x_0

$$u(x_0) := \lim_{\varrho\to 0} (u)_{B_\varrho(x_0)} = \lim_{\varrho\to 0} \fint_{B_\varrho(x_0)} u\,dx$$

and the following estimate holds

$$|u(x_0)| \leq c\mathbf{W}^{\mu}_{\alpha,p}(x_0, r) + c\left(\fint_{B_r(x_0)} |u|^{p-1}\,dx\right)^{1/p-1} + c\mathrm{Tail}(u; x_0, r)$$

Comparison with the local case

In the case $-\mathrm{div}\,(|Du|^{p-2}Du) = \mu$ we have

$$|u(x)| \lesssim \mathbf{W}^{\mu}_{1,p}(x,R) + \left(\fint_{B_R(x)} |u|^{p-1}\,dy\right)^{1/(p-1)}$$

where

$$\mathbf{W}^{\mu}_{1,p}(x,R) := \int_0^R \left(\frac{|\mu|(B_\varrho(x))}{\varrho^{n-p}}\right)^{1/(p-1)} \frac{d\varrho}{\varrho}$$

In the fractional case we use

$$\mathbf{W}^{\mu}_{\alpha,p}(x,R) := \int_0^R \left(\frac{|\mu|(B_\varrho(x))}{\varrho^{n-p\alpha}}\right)^{1/(p-1)} \frac{d\varrho}{\varrho}$$

Wolff potentials bounds

Theorem (Kuusi & Min. & Sire, CMP 15)

Let $\mu \in \mathcal{M}(\mathbb{R}^n)$, $g \in W^{\alpha,p}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^{p-1}_{p\alpha}(\mathbb{R}^n)$. *Let* u *be a SOLA. If*

$$\lim_{t\to 0} \sup_{x\in\Omega'} \mathbf{W}^{\mu}_{\alpha,p}(x,t) = 0\,,$$

then u *is continuous in* Ω'. *In particular, this happens if*

$$\mu \in L\left(\frac{n}{p\alpha}, \frac{1}{p-1}\right) \qquad \textit{with} \quad p\alpha < n$$

or

$$\mu \in L^q\,, \qquad q > \frac{n}{p\alpha}\,.$$

Classical self-improving properties

Theorem (Elcrat-Meyers, Giaquinta-Modica)

Let u be a weak solution to

$$-\operatorname{div} a(x, Du) = f \in L^{2+\delta_0}$$

where

$$\frac{|z|^2}{\Lambda} \leq \langle a(x,z), z\rangle \qquad \textit{and} \qquad |a(x,z)| \leq \Lambda |z|$$

Classical self-improving properties

Theorem (Elcrat-Meyers, Giaquinta-Modica)

Let u be a weak solution to

$$-\mathrm{div}\, a(x, Du) = f \in L^{2+\delta_0}$$

where

$$\frac{|z|^2}{\Lambda} \leq \langle a(x,z), z\rangle \qquad \textit{and} \qquad |a(x,z)| \leq \Lambda |z|$$

Then

$$u \in W^{1,2} \Longrightarrow u \in W^{1,2+\delta}_{\mathrm{loc}}$$

for some $\delta > 0$ depending only on n, Λ, δ_0

The Gehring lemma with additional terms

Theorem (Gehring-Giaquinta-Modica)

Let $f \in L^p_{\mathrm{loc}}(\Omega)$ be such that

$$\left(\fint_{B/2} f^p\,dx\right)^{1/p} \lesssim \left(\fint_B f^q\,dx\right)^{1/q} + \left(\fint_B g^p\,dx\right)^{1/p}$$

for $q < p$, then

$$\left(\fint_{B/2} f^{p+\delta}\,dx\right)^{1/(p+\delta)} \lesssim \left(\fint_B f^q\,dx\right)^{1/q} + \left(\fint_B g^{p+\delta}\,dx\right)^{1/(p+\delta)}$$

Caccioppoli inequalities imply higher integrability

Theorem

Let $u \in W^{1,2}(\mathbb{R}^n)$ such that for every ball $B \equiv B(x_0, r) \subset \mathbb{R}^n$

$$\fint_{B/2} |Du|^2 dx \lesssim \frac{1}{r^2} \fint_B |u(x) - (u)_B|^2 \, dx$$

holds; then there exists $\delta > 0$ such that

$$u \in W^{1,2+\delta}_{\text{loc}}(\mathbb{R}^n)$$

The proof is very simple: Sobolev-Poincaré yields

$$\left(\fint_{B/2} |Du|^2 dx\right)^{1/2} \lesssim \left(\fint_{B} |Du|^{2n/(n+2)} dx\right)^{(n+2)/2n}$$

and the assertion follows from Gehring lemma

No gradient oscillations control

Consider

$$(a(x)u_x)_x = 0$$

with

$$0 < \nu \leq a(x) \leq L$$

then

$$x \mapsto \int^x \frac{dt}{a(t)}$$

i.e. no gradient differentiability is possible when coefficients are just differentiable

Integrodifferential equations

We consider

$$\mathcal{E}_K(u,\varphi) = \int_{\mathbb{R}^n} f\varphi\, dx$$

For every test function $\varphi \in C_0^\infty(\mathbb{R}^n)$ where

$$\mathcal{E}_K(u,\varphi) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} [u(x)-u(y)][\varphi(x)-\varphi(y)]K(x,y)\, dx\, dy$$

The Kernel satisfies

$$\frac{1}{\Lambda|x-y|^{n+2\alpha}} \leq K(x,y) \leq \frac{\Lambda}{|x-y|^{n+2\alpha}}$$

for some $\Lambda \geq 1$

Energy solutions are initially considered in

$$u \in W^{\alpha,2}(\mathbb{R}^n)$$

The analogue of the Meyers property is now

$$u \in W^{\alpha,2+\delta}, \qquad \delta > 0$$

upon considering $f \in L^q$ for some $q > 2$

For $\alpha \in (0,1)$

$$[u]^2_{\alpha,2} := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy$$

The usual gradient can be obtained letting $\alpha \to 1$, but only after renormalisation, see the work of Bourgain & Brezis & Mironescu

Self-improving property

Theorem (Kuusi & Min. & Sire, Analysis & PDE 15)

$$u \in W^{\alpha+\delta,2+\delta} \qquad \textit{for some } \delta > 0$$

Self-improving property

Theorem (Kuusi & Min. & Sire, Analysis & PDE 15)

$$u \in W^{\alpha+\delta,2+\delta} \qquad \textit{for some } \delta > 0$$

This theorem has no analog in the local, classical case, where the improvement is only in the integrability scale

$$u \in W^{1,2+\delta}_{\mathrm{loc}}$$

Fractional improvement

Theorem (Schikorra, Math. Ann., to appear)

There exists a number

$$\delta_0 \equiv \delta_0(n, \alpha, \Lambda)$$

such that any $W^{\alpha-\delta_0, 2-\delta_0}$*-solution* u *to the equation*

$$\mathcal{E}_K(u, \varphi) = 0$$

is such that

$$u \in W^{\alpha+\delta_0, 2+\delta_0}_{\mathrm{loc}}(\mathbb{R}^n)$$

This extends to the nonlocal p-Laplacean as well

A fractional approach to Gehring lemma

Caccioppoli inequalities imply higher integrability - local case

Theorem

Let $u \in W^{1,2}(\mathbb{R}^n)$ such that for every ball $B \equiv B(x_0, r) \subset \mathbb{R}^n$

$$\fint_{B/2} |Du|^2 dx \lesssim \frac{1}{r^2} \fint_B |u(x) - (u)_B|^2 \, dx$$

holds; then there exists $\delta > 0$ such that

$$u \in W^{1,2+\delta}_{\text{loc}}(\mathbb{R}^n)$$

Caccioppoli inequalities imply higher integrability - nonlocal case

Theorem (Kuusi & Min. & Sire, Analysis & PDE 15)

Let $u \in W^{\alpha,2}(\mathbb{R}^n)$ *such that for every ball* $B \equiv B(x_0, r) \subset \mathbb{R}^n$

$$\fint_B \fint_B \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}\, dx\, dy \lesssim \frac{1}{r^{2\alpha}} \fint_B |u(x)-(u)_B|^2\, dx + \int_{\mathbb{R}^n \setminus B} \frac{|u(y)-(u)_B|}{|x_0-y|^{n+2\alpha}}\, dy \fint_B |u(x)-(u)_B|\, dx$$

holds; then there exists $\delta > 0$ *such that*

$$u \in W^{\alpha+\delta,2+\delta}_{\mathrm{loc}}(\mathbb{R}^n)$$

Key observation

- $u \in W^{1,2}$ means that $|Du|^2$ is integrable w.r.t. a **finite** measure (i.e. the Lebesgue measure)

Key observation

- $u \in W^{1,2}$ means that $|Du|^2$ is integrable w.r.t. a **finite** measure (i.e. the Lebesgue measure)
- $u \in W^{\alpha,2}$ means that

$$\left[\frac{|u(x) - u(y)|}{|x - y|^{\alpha}}\right]^2$$

is integrable w.r.t. an **infinite** set function, that is

$$E \to \int_E \frac{dx\, dy}{|x - y|^n}$$

there are therefore potentially more regularity properties to exploit in the above fractional difference quotient

Key idea: Dual pairs

To each u in $\mathbb{R}^{2n}$ and $\varepsilon < (0, 1-\alpha)$ we associate a function

$$U(x,y) := \frac{|u(x)-u(y)|}{|x-y|^{\alpha+\varepsilon}}$$

and a **doubling** measure

$$\mu(E) := \int_E \frac{dx\,dy}{|x-y|^{n-2\varepsilon}}$$

and note that they are in duality in the sense that

$$u \in W^{\alpha,2} \Longleftrightarrow U \in L^2(\mu)$$

Strategy: higher integrability for U w.r.t. μ

- We translate the Caccioppoli inequality for u in a reverse Hölder inequality for U w.r.t. μ
- We prove a version of Gehring lemma for dual pairs (μ, U)
- The higher integrability of U turns into the higher differentiability of u
- All estimates heavily degenerate when $\alpha \to 1$ or $\alpha \to 0$

Assume $U \in L^{2+\delta}_{\text{loc}}$, this means that

$$\int_{B\times B} U^{2+\delta}\, d\mu = \int_B \int_B \frac{|u(x)-u(y)|^{2+\delta}}{|x-y|^{n+(2+\delta)\alpha+\varepsilon\delta}}\, dx\, dy < \infty$$

rewrite as follows:

$$\int_B \int_B \frac{|u(x)-u(y)|^{2+\delta}}{|x-y|^{n+(2+\delta)[\alpha+\varepsilon\delta/(2+\delta)]}}\, dx\, dy < \infty$$

and this means that

$$u \in W^{\alpha+\varepsilon\delta/(2+\delta),2+\delta}_{\text{loc}}(\mathbb{R}^n)$$

i.e. we have gained differentiability

The Gehring lemma for dual pairs (μ, U)

Theorem (Kuusi & Min. & Sire, Analysis & PDE 15)

If (μ, U) *satisfiers*

$$\left(\fint_{\mathcal{B}} U^2 d\mu\right)^{1/2} \leq c(\sigma)\left(\fint_{\mathcal{B}} U^q\, d\mu\right)^{1/q} + \sigma \sum_{k=2}^{\infty} 2^{-k(\alpha-\varepsilon)}\left(\fint_{2^k\mathcal{B}} U^q\, d\mu\right)^{1/q}$$

where $q \in (1,2)$ *and for every choice of* $\mathcal{B} = B \times B$, *then*

$$U \in L^{2+\delta}_{\mathrm{loc}} \qquad \textit{for some} \ \ \delta > 0$$

and

$$\left(\fint_{\mathcal{B}} U^{2+\delta}\, d\mu\right)^{1/(2+\delta)} \lesssim \sum_{k=1}^{\infty} 2^{-k(\alpha-\varepsilon)}\left(\fint_{2^k\mathcal{B}} U^2\, d\mu\right)^{1/2}$$