Homogeneous Boltzmann equation and the α-stable processes

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Joint work with **Marco Cannone**
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The anomalous diffusion equation with $\alpha \in (0,2]$

$$\partial_t u + (-\Delta)^{\alpha/2} u = 0.$$

In the Fourier variables:

$$\partial_t \widehat{u}(\xi,t) + |\xi|^{\alpha} \widehat{u}(\xi,t) = 0.$$

Initial condition:

$$\widehat{u}(\xi,0) = \widehat{u}_0(\xi)$$

The solution:

$$\widehat{u}(\xi,t) = \widehat{u}_0(\xi) e^{-t|\xi|^{\alpha}}$$

The self-similar solution:

$$\widehat{p}_{\alpha}(\xi,t) = e^{-t|\xi|^{\alpha}} = e^{-|\xi t^{\frac{1}{\alpha}}|^{\alpha}}$$

It is well-known that:

$$p_{\alpha}(x,t) = t^{-\frac{d}{\alpha}} P_{\alpha}(x t^{-\frac{1}{\alpha}})$$

The anomalous diffusion equation with $\alpha \in (0,2]$

$$\partial_t u + (-\Delta)^{\alpha/2} u = 0.$$

Scaling

$$u_B(x,t) = B^d u(Bx, B^\alpha t) \qquad \text{for all } B > 0.$$

In the Fourier variables

$$\widehat{u}_B(\xi,t) = \widehat{u}(\xi/B, B^\alpha t) \qquad \text{for all } B > 0.$$

Self-similarity: For $p_\alpha(x,t) = t^{-\frac{d}{\alpha}} P_\alpha(xt^{-\frac{1}{\alpha}})$, we have

$$\big(p_\alpha(x,t)\big)_B = p_\alpha(x,t)$$

Central Limit Theorem

Theorem

Assume that $u(x,t)$ is a solution of

$$\partial_t u + (-\Delta)^{\alpha/2} u = 0, \qquad u(x,0) = u_0(x)$$

with $u_0 \in L^1(\mathbb{R}^d)$.

Then

$$u_B(x,t) \to p_\alpha(x,t) \int_{\mathbb{R}^d} u_0(x)\, dx \qquad \text{as } B \to \infty.$$

Proof.

$$\widehat{u}_B(\xi,t) = \widehat{u}_0(\xi/B)\widehat{p}_\alpha(\xi/B, tB^\alpha) \to \widehat{u}_0(0)\widehat{p}_\alpha(\xi,t)$$

as $B \to \infty$. □

Remark

For $B = t^{1/\alpha}$, we have

$$u_B(x,1) = B^d u(Bx, B^\alpha) = t^{d/\alpha} u(xt^{1/\alpha}, t).$$

The Boltzmann equation in $\mathbb{R}^3$

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f,f)$$
$$f(x,v,0) = f_0(x,v)$$

where

$$f = f(x,v,t) \qquad \text{with} \quad (x,v,t) \in \mathbb{R}^3 \times \mathbb{R}^3 \times (0,\infty)$$

and $Q(f,f)$ is the collision term, f_0 the initial data and f the unknown.

Homogeneous Boltzmann equation:

$$f = f(v,t) \quad \text{independent of } x$$

Homogeneous Boltzmann equation in $\mathbb{R}^3$

$$\partial_t f(v,t) = Q(f,f)(v,t)$$

with the bilinear form corresponding to a Maxwellian gas

$$Q(g,f)(v) = \int_{\mathbb{R}^3} \int_{S^2} \mathcal{B}\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right)\big(f(v')g(v'_*) - f(v)g(v_*)\big)\, d\sigma\, dv_*.$$

where

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma$$

with σ varying in the unit sphere S^2.

- The collision kernel $\mathcal{B}$ is a nonnegative function.
- In the case of Maxwellian molecules, it depends only on the deviation angle θ, defined by the equation $\cos\theta = \frac{v-v_*}{|v-v_*|}\cdot\sigma$.
- $\mathcal{B} = \mathcal{B}(s)$ has a nonintegrable singularity as $s \to 1$ of the form $(1-s)^{-5/4}$

Finite energy solutions

It is natural to assume that the nonnegative initial datum satisfies

$$\int_{\mathbb{R}^3} f_0(v)\,dv = 1, \quad \int_{\mathbb{R}^3} f_0(v) v_i\,dv = 0\ (i = 1, 2, 3), \quad \int_{\mathbb{R}^3} f_0(v)|v_i|^2\,dv = 1,$$

the unit mass,
the zero mean value,
the unit temperature of the gas

For such initial data, the existence of a unique solution

$$f \in C^1([0, \infty), L^1(\mathbb{R}^3))$$

of the initial value problem for the homogeneous Boltzmann equation (not necessarily for Maxwellian molecules).

This solution satisfies

$$\int_{\mathbb{R}^3} f(v,t)\,dv = 1, \quad \int_{\mathbb{R}^3} f(v,t) v_i\,dv = 0\ (i = 1, 2, 3), \quad \int_{\mathbb{R}^3} f(v,t)|v_i|^2\,dv = 1$$

for all $t > 0$.

The Bobylev formulation

The Fourier transform

$$\varphi(\xi, t) \equiv \widehat{f}(\xi, t) = \int_{\mathbb{R}^3} e^{-iv\cdot\xi} f(v, t)\, dv$$

The homogeneous Boltzmann equation for Maxwellian molecules:

$$\partial_t \varphi(\xi, t) = \int_{S^2} \mathcal{B}\left(\frac{\xi \cdot \sigma}{|\xi|}\right) \big(\varphi(\xi^+, t)\varphi(\xi^-, t) - \varphi(\xi, t)\varphi(0, t)\big)\, d\sigma$$

where

$$\xi^+ = \frac{\xi + |\xi|\sigma}{2}, \quad \xi^- = \frac{\xi - |\xi|\sigma}{2}$$

These two vectors ξ^+ and ξ^- satisfy the well-known relations

$$\xi^+ + \xi^- = \xi \quad \text{and} \quad |\xi^+|^2 + |\xi^-|^2 = |\xi|^2.$$

The Cauchy problem

The equation

$$\partial_t \varphi(\xi, t) = \int_{S^2} \mathcal{B}\left(\frac{\xi \cdot \sigma}{|\xi|}\right) \big(\varphi(\xi^+, t)\varphi(\xi^-, t) - \varphi(\xi, t)\varphi(0, t)\big)\, d\sigma$$

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Initial condition

$$\varphi(\xi, 0) = \varphi(\xi)$$

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Explicit stationary solutions:

- $\varphi \equiv 1$ (Dirac measure)
- $\varphi = e^{-K|\xi|^2}$ (Maxwellian), because $|\xi^+|^2 + |\xi^-|^2 = |\xi|^2$.

Continuous positive definite functions

A function $\varphi : \mathbb{R}^N \to \mathbb{C}$ is called *a characteristic function* if there is a probability measure μ (i.e. a Borel measure with $\int_{\mathbb{R}^N} \mu(dx) = 1$) such that we have the following identity

$$\varphi(\xi) = \widehat{\mu}(\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi}\, \mu(dx).$$

Simple property: φ is continuous and bounded

$$|\varphi(\xi)| \leqslant \varphi(0) = 1$$

Existence under the cut-off assumption

The pseudo-Maxwellian gas:

$$\int_{S^2} \mathcal{B}\left(\frac{\xi\cdot\sigma}{|\xi|}\right) d\sigma \quad \text{is finite for all} \quad \xi \in \mathbb{R}^3 \setminus \{0\}.$$

The initial value problem:

$$\partial_t \varphi(\xi, t) = \int_{S^2} \mathcal{B}\left(\frac{\xi\cdot\sigma}{|\xi|}\right) \big(\varphi(\xi^+, t)\varphi(\xi^-, t) - \varphi(\xi, t)\varphi(0, t)\big)\, d\sigma$$

$$\varphi(\xi, 0) = \varphi_0(\xi)$$

The Banach contraction principle gives global-in-time unique solutions in the space $C([0, T], C_b(\mathbb{R}^3))$ for all $T > 0$.
This is a characteristic function for every t.

Toscani distance

For each $\alpha \in [0,2]$, we define

$$\mathcal{K}^\alpha = \Big\{\varphi : \mathbb{R}^3 \to \mathbb{C} \text{ is a characteristic function such that } \|\varphi - 1\|_\alpha < \infty\Big\},$$

where

$$\|\varphi - 1\|_\alpha \equiv \sup_{\xi \in \mathbb{R}^3} \frac{|\varphi(\xi) - 1|}{|\xi|^\alpha}.$$

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PROPERTIES

- The set $\mathcal{K}^\alpha$ endowed with the **Toscani distance**

$$\|\varphi - \widetilde{\varphi}\|_\alpha \equiv \sup_{\xi \in \mathbb{R}^3} \frac{|\varphi(\xi) - \widetilde{\varphi}(\xi)|}{|\xi|^\alpha}.$$

  is a complete metric space.

- The following imbeddings hold true

$$\{1\} \subseteq \mathcal{K}^\alpha \subseteq \mathcal{K}^{\alpha_0} \subseteq \mathcal{K}^0 \quad \text{for all} \quad 2 \geqslant \alpha \geqslant \alpha_0 \geqslant 0.$$

$$\|\varphi - 1\|_\alpha \equiv \sup_{\xi \in \mathbb{R}^3} \frac{|\varphi(\xi) - 1|}{|\xi|^\alpha}.$$

Lemma

If

$$\int_{\mathbb{R}^3} f_0(v)\, dv = 1, \quad \int_{\mathbb{R}^3} f_0(v) v_i\, dv = 0 \quad \int_{\mathbb{R}^3} f_0(v) |v_i|^2\, dv = 1,$$

then

$$\varphi_0 = \widehat{f_0} \in \mathcal{K}^2.$$

$$\|\varphi - 1\|_\alpha \equiv \sup_{\xi \in \mathbb{R}^3} \frac{|\varphi(\xi) - 1|}{|\xi|^\alpha}.$$

EXAMPLES

- Maxwellians in the Fourier variables, $\varphi(\xi) = e^{-A|\xi|^2}$ with fixed $A > 0$, belongs to $\mathcal{K}^\alpha$ for every $\alpha \in [0, 2]$.

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- The Fourier transform of any probability measure with the finite moment of order α belongs to $\mathcal{K}^\alpha$.

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- Maxwellians in the Fourier variables, $\varphi(\xi) = e^{-A|\xi|^2}$ with fixed $A > 0$, belongs to $\mathcal{K}^\alpha$ for every $\alpha \in [0, 2]$.

- The Fourier transform of any probability measure with the finite moment of order α belongs to $\mathcal{K}^\alpha$.

- For each $\alpha \in (0, 2]$

$$\varphi_\alpha(\xi) = e^{-|\xi|^\alpha} \in \mathcal{K}^\alpha.$$

  Its inverse Fourier transform has infinite moment of order α.

Fundamental estimates

Lemma

For any positive definite function $\varphi = \varphi(\xi)$ such that $\varphi(0) = 1$ we have

$$|\varphi(\xi) - \varphi(\eta)|^2 \leqslant 2\big(1 - \Re\varphi(\xi - \eta)\big)$$

and

$$|\varphi(\xi)\varphi(\eta) - \varphi(\xi + \eta)|^2 \leqslant (1 - |\varphi(\xi)|^2)(1 - |\varphi(\eta)|^2)$$

for all $\xi, \eta \in \mathbb{R}^N$.

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Lemma

Let $\alpha \in [0, 2]$. Assume that $\varphi \in \mathcal{K}^\alpha$. Then

$$|\varphi(\xi^+)\varphi(\xi^-) - \varphi(\xi)| \leqslant 4|\xi^+|^{\alpha/2}|\xi^-|^{\alpha/2}\|\varphi - 1\|_\alpha.$$

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$$|\varphi(\xi^+)\varphi(\xi^-) - \varphi(\xi)| \leqslant 4|\xi^+|^{\alpha/2}|\xi^-|^{\alpha/2}\|\varphi - 1\|_\alpha.$$

Thus, for $\varphi \in \mathcal{K}^\alpha$, the nonlinear term

$$\int_{S^2} \mathcal{B}\left(\frac{\xi \cdot \sigma}{|\xi|}\right) \big(\varphi(\xi^+)\varphi(\xi^-) - \varphi(\xi)\varphi(0)\big)\, d\sigma$$

is well-defined for some singular $\mathcal{B}$.

Existence for singular kernels

Standing assumption

$$(1-s)^{\alpha_0/4}(1+s)^{\alpha_0/4}\mathcal{B}(s) \in L^1(-1,1) \quad \text{for some} \quad \alpha_0 \in [0,2].$$

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Theorem ((Cannone & K. (2010))

For each $\alpha \in [\alpha_0, 2]$ and every $\varphi_0 \in \mathcal{K}^\alpha$ there exists a classical solution

$$\varphi \in C([0,\infty), \mathcal{K}^\alpha)$$

of our problem

The solution is unique in the space $C([0,\infty), \mathcal{K}^{\alpha_0})$.

...

Recent improvements by **Morimoto, Wang, & Yang**.

Stability of solutions

Theorem ((Cannone & K. (2010))

Assume that $\mathcal{B}$ satisfies the standing assumption.

Let $\alpha \in [\alpha_0, 2]$ and consider two solutions $\varphi, \widetilde{\varphi} \in C([0, \infty, \mathcal{K}^\alpha)$ of our problem corresponding to the initial data $\varphi_0, \widetilde{\varphi}_0 \in \mathcal{K}^\alpha$, respectively.

Then for every $t \geqslant 0$

$$\|\varphi(t) - \widetilde{\varphi}(t)\|_\alpha \leqslant e^{\lambda_\alpha t} \|\varphi_0 - \widetilde{\varphi}_0\|_\alpha,$$

where the constant $\lambda_\alpha \geqslant 0$ is defined by

$$\lambda_\alpha \equiv \int_{S^2} \mathcal{B}\left(\frac{\xi \cdot \sigma}{|\xi|}\right) \left(\frac{|\xi^-|^\alpha + |\xi^+|^\alpha}{|\xi|^\alpha} - 1\right) d\sigma.$$

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...

Proof.
The Gronwall lemma and approximations. □

Propagation of generalized moments (Cannone, K. & Ntovoris (2016))

Propagation of generalized moments (Cannone, K. & Ntovoris (2016))

We say that a constant $K > 0$ is an (isotropic) generalized α-moment of a function $\varphi \in \mathcal{K}^\alpha$ if

$$\lim_{\xi \to 0} \frac{1 - \varphi(\xi)}{|\xi|^\alpha} = K$$

provided this limit exists.

Theorem

Assume that a collision kernel $\mathcal{B}$ satisfies condition non-cutoff conditions for some $\alpha_0 \in [0, 2)$. Consider a solution $\varphi \in C([0, +\infty), \mathcal{K}^\alpha)$ with certain $\alpha \in [\alpha_0, 2)$. Suppose that

$$\lim_{\xi \to 0} \frac{1 - \varphi(\xi, 0)}{|\xi|^\alpha} = K \qquad \textit{for some} \quad K > 0.$$

Then

$$\lim_{\xi \to 0} \frac{1 - \varphi(\xi, t)}{|\xi|^\alpha} = K e^{\lambda_\alpha t} \qquad \textit{for all} \quad t \geqslant 0.$$

Propagation of generalized moments (Cannone, K. & Ntovoris (2016))

$$\lim_{\xi \to 0} \frac{1 - \varphi(\xi, t)}{|\xi|^\alpha} = Ke^{\lambda_\alpha t} \qquad \text{for all} \quad t \geqslant 0.$$

For $\alpha = 2$, we have $\lambda_\alpha = 0$. Hence, our formula generalizes classical conservation laws.

Self-similar eternal solutions (Cannone, K. & Ntovoris (2016))

Theorem

Fix $w(\cdot, 1) \in \mathcal{K}^\alpha$ *such that*

$$\lim_{\xi \to 0} \frac{1 - w(\xi, 1)}{|\xi|^\alpha} = K \qquad \text{for some } K > 0.$$

(for example: $w(\xi, 1) = e^{-K|\xi|^\alpha}$*.)*

Let $w(\xi, s)$ *be the corresponding solution of the (slowed) problem.*

Then

$$w_B(\xi, s) = w(\xi B^{-\mu_\alpha}, sB) \to \bar{w}(\xi, s)$$

as $B \to \infty$*.*

Self-similar eternal solutions (Cannone, K. & Ntovoris (2016))

Theorem

The limit function $\bar{w}$ has the following properties:

- $\bar{w}$ *is a solution of the (slowed) equation on* $(0, \infty)$*;*
- $\bar{w} \in C((0, \infty), \mathcal{K}^\alpha)$ *and* $|||w - 1|||_\alpha < \infty$*;*
- *Evolution of "α-moment":*

$$\lim_{\xi \to 0} \frac{1 - \bar{w}(\xi, s)}{|\xi|^\alpha} = K s^{\lambda_\alpha}$$

  for all $s > 0$*.*

- $\bar{w}$ *is self-similar:*

$$\bar{w}(\xi, s) = \bar{w}(\xi s^{\mu_\alpha}, 1).$$

Back to original variables

Solution of the (slowed) equation:

$$\bar{w}(\xi, s) = \bar{w}(\xi s^{\mu_\alpha}, 1).$$

Solution of the Boltzmann equation in the Fourier variables:

$$\bar{\varphi}(\xi, t) = \bar{\varphi}(\xi e^{\mu_\alpha t}, 1).$$

Solution of the homogeneous Boltzmann equation for Maxwellian molecules

$$\bar{f}(v, t) = e^{-3\mu_\alpha t} \bar{f}(v e^{-\mu_\alpha t}, 1) = e^{-3\mu_\alpha t} F(v e^{-\mu_\alpha t}).$$

These are Bobylev & Cercignani (2002) self-similar solutions

Self-similar solutions of the Boltzmann equation

$$\bar{f}(v,t) = e^{-3\mu_\alpha t} F(ve^{-\mu_\alpha t}).$$

Properties of the profile F:

- F depends on $K > 0$; $F > 0$; $\int\limits_{\mathbb{R}^3} F(v)\, dv = 1$.
- If $\alpha = 2$, then $\mu_\alpha = 0$ and F is Maxwellian: $\widehat{F}(\xi) = e^{-K|\xi|^2}$.
- For $\alpha \in (0,2)$, we have $F \in C_b^\infty(\mathbb{R}^3)$.
- It has moments:

$$\int\limits_{\mathbb{R}^3} F(v)|v|^\alpha\, dv = \infty \quad \text{and} \quad \int\limits_{\mathbb{R}^3} F(v)|v|^\beta\, dv < \infty \quad \text{for each } \beta \in (0,\alpha).$$

Central Limit Theorem

Theorem

Let $\alpha \in (0, 2)$.

If

$$\int_{\mathbb{R}^3} |f_0(v) - F(v)|\, |v|^\alpha \, dv < \infty$$

then

$$e^{3\mu_\alpha t} f\left(e^{\mu_\alpha t} v, t\right) \to F(v) \qquad as \quad t \to \infty,$$

weakly.

On the other hand, if $\int_{\mathbb{R}^3} f_0(v)|v|^\alpha \, dv < \infty$, *then*

$$e^{3\mu_\alpha t} f\left(e^{\mu_\alpha t} v, t\right) \to \delta_0 \qquad as \quad t \to \infty,$$

weakly.