Further Time Regularity for Parabolic Equations

Joint with Dennis Kriventsov

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Linear elliptic operators of order $\sigma \in (0,2)$

For $K : \mathbb{R}^n \to [\lambda, \Lambda] \subseteq (0,\infty)$ and even $(K(y) = K(-y))$

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For $K \equiv 1$,

$$\Delta^{\sigma/2} u(x) = c_{n,\sigma} \int (u(x+y) - u(x)) \frac{dy}{|y|^{n+\sigma}} \sim \Delta u(x)$$

Nonlinear elliptic operators of order $\sigma \in (0,2)$

I applied to $u : \mathbb{R}^n \to \mathbb{R}$ gives $Iu : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ such that $Iu = 0$ if $u \equiv constant$ $\sim F(D^2u, x, t)$ where $F(0, x, t) = 0$

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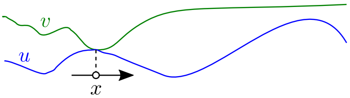

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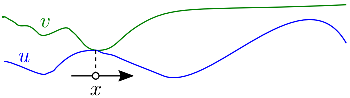

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Uniform Ellipticity: For $0 < \lambda \leq \Lambda < \infty$

$$\inf_{\substack{K\in[\lambda,\Lambda]\\ \text{even}}} L(u-v) \leq Iu - Iv \leq \sup_{\substack{K\in[\lambda,\Lambda]\\ \text{even}}} L(u-v)$$

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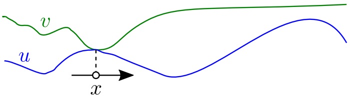

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In particular, uniform ellipticity $\Rightarrow$ ellipticity.

Examples

- **Linear operators with variable coefficients:**
  $\forall x \in \Omega, K(\cdot, x) : \mathbb{R}^n \to [\lambda, \Lambda] \subseteq (0, \infty)$ even

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- **Extremal operators:**

$$\mathcal{M}_\sigma^+ u = \sup_{\substack{K \in [\lambda,\Lambda] \\ \text{even}}} Lu \qquad \text{and} \qquad \mathcal{M}_\sigma^- u = \inf_{\substack{K \in [\lambda,\Lambda] \\ \text{even}}} Lu$$

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- **Inf/sup combinations:**

$$Iu = \inf_\alpha \sup_\beta L_{\alpha,\beta} u \qquad K_{\alpha,\beta}(y) \in [\lambda, \Lambda] \text{ even}$$

Initial-boundary value problem

Let u solve

$$\begin{aligned} \partial_t u - Iu = f(x,t) &\quad \text{in} \quad Q_1 = B_1 \times (-1,0] \\ u = g &\quad \text{on} \quad (\mathbb{R}^n \setminus B_1) \times (-1,0] \\ u = u_0 &\quad \text{on} \quad t = -1 \end{aligned}$$

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How do the data influence the regularity of u?

Regularity theory for $\partial_t u - F(D^2u, x, t) = f(x, t)$

- Krylov-Safonov (1979): For $\lambda Id \leq (a_{i,j}(x,t)) \leq \Lambda Id$ symmetric

$$\partial_t u - \sum a_{ij}(x,t)\partial_{ij} u \in L^\infty(Q_1) \quad \Rightarrow \quad u \in C^\alpha(Q_{1/2})$$

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- For $F(D^2u, x, t) = F(D^2u)$ translation invariant

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- Evans-Krylov (1982): For F translation invariant and concave

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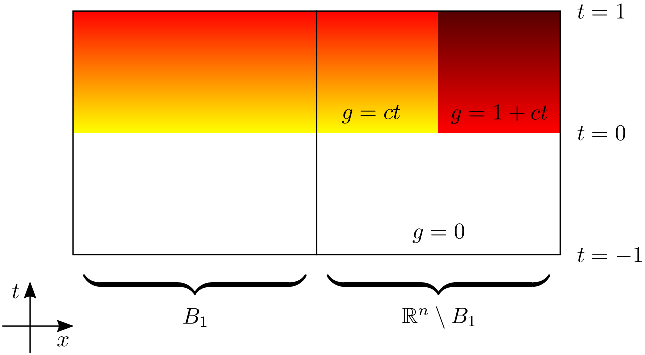
Is it true that as $\sigma \to 2$, $\partial_t u$ is Hölder continuous?

Answer: No!

$$\partial_t u = \Delta^{\sigma/2} u$$



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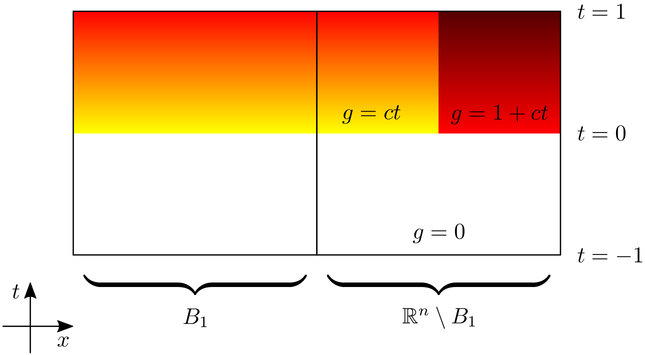

We need a hypothesis for g!

Main Result: Hölder estimate for $\partial_t u$

Theorem (ChL-Kriventsov, CPAM, to appear)

Let $\sigma \in [1,2)$, I be translation invariant and $u \in L^\infty(\mathbb{R}^n \times (-1,0])$ solve

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We only require I translation invariant in time and

$$\sup_{t\in[-1,0]} \int |u(y)|(1 \wedge |y|^{-(n+\sigma)})dy < \infty$$

$$\sup_{x\in B_1} [f(x,\cdot)]_{C^{\gamma/\sigma}((-1,0])} < \infty$$

$$\sup_{(t-\tau,t]\subseteq(-1,0]} \int_{\mathbb{R}^n\setminus B_1} \frac{|g(y,t-\tau) - g(y,t)|}{\tau^{\gamma/\sigma}} \frac{dy}{|y|^{n+\sigma}} < \infty$$

Remarks about the corresponding local result

Corresponding **local** estimate seemed to be unknown.

Theorem (ChL-Kriventsov, Trans. Math. Res. Let., to appear)

Let u solve

$$\partial_t u - F(D^2u, x) = f(x,t) \text{ in } Q_1$$

Then, for $\gamma \in (0,\alpha)$ ($\alpha \in (0,1)$ from Krylov-Safonov)

$$[\partial_t u]_{C^\gamma(Q_{1/2})} \leq C\left(\|u\|_{L^\infty(Q_1)} + \sup_{x\in B_1}[f(x,\cdot)]_{C^{\gamma/2}(-1,0]}\right)$$

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- $F(D^2u) \in C^\gamma$ does not contradict the counterexamples of Nadirashvili-Vlăduţ ($D^2u \notin L^\infty$ for some F)
- Scaling for f corresponds to the scaling for $\partial_t u$
- $f(x,t) = f(t)$ is trivial (Consider $v = u - \int f dt$...)

Review of the $C^{1,\alpha}$ regularity for $\partial_t u - F(D^2u) = 0$

Let $\partial_t u - F(D^2u) = 0$ in Q_1

1. From $F(0) = 0$

$$\partial_t u = F(D^2u) = \sum \underbrace{a_{ij}(x,t)}_{=\int_0^1 \partial_{m_{ij}} F(sD^2u)ds} \partial_{ij} u \text{ in } Q_1$$

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2. By Krylov-Safonov $u \in C^\alpha(Q_{1/2})$ so

$$u_{\tau,\alpha} = \frac{\delta_\tau u}{\tau^\alpha} = \frac{u_\tau - u}{\tau} = \frac{u(t-\tau) - u(t)}{\tau} \in L^\infty(Q_{1/4})$$

for any $\tau \in (0, 1/4)$

Review of the $C^{1,\alpha}$ regularity for $\partial_t u - F(D^2u) = 0$

3. From the translation invariance and the uniform elipticity

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$$u_{\tau,\alpha} \in C^\alpha(Q_{1/8}) \underset{\text{interpolation}}{\Rightarrow} u_{\tau,2\alpha} \in L^\infty(Q_{1/16})$$

uniformly in $\tau \in (0, 1/16)$

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   uniformly in $\tau \in (0, 1/16)$

5. Iterate $(k+1)$ times until $k\alpha < 1 < (k+1)\alpha$. By one final interpolation at the last step we conclude that

$$\partial_t u \in C^\alpha(Q_{1/4^{k+1}})$$

What fails in the nonlocal setting?

Short answer: The C^α estimate depends on the size of the tail.

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For $K(y;x,t) \in [\lambda, \Lambda]$ *even in* y

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In particular,

$$u \in L^\infty(\mathbb{R}^n \times (-1,0]) \Rightarrow u_{\tau,\alpha} \in L^\infty(Q_{1/4})$$

but not in $L^\infty(\mathbb{R}^n \times (-1/4,0])$ as required in the subsequent step.

Boundary data → Right-hand side

Standard trick: Multiply u (or $u_{\tau,i\alpha}$) by $\eta \in C_0^\infty(B_r)$ such that $\eta = 1$ in $B_{r/2}$.

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- This truncation is applied at every step to go from $u_{\tau,i\alpha} \in L^\infty(Q_r)$ to $u_{\tau,(i+1)\alpha} \in L^\infty(Q_{r/4})$.

Hölder Bootstrap

Starting from

$$\partial_t u_{\tau,\beta} - L u_{\tau,\beta} = \frac{\delta_\tau f}{\tau^\beta} \text{ in } Q_1 \text{ and for all } \tau$$

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$$\begin{aligned} \|u_{\tau,\beta}\|_{L^\infty} \leq 1 \qquad &\Rightarrow \qquad [u_{\tau,\beta}]_{C^\alpha} \leq C \\ &\Rightarrow \qquad \|u_{\tau,\beta+\alpha}\|_{L^\infty} \leq C \end{aligned}$$

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But... as $\tau \to 0$ the right-hand side degenerates because f is only in C^γ, and γ/σ could be less than β.

Hölder Bootstrap with stroger hypotheis

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What we managed to do:

$$\sup_\tau [u_{\tau,\beta}]_{C^\varepsilon} \leq 1 \quad \Rightarrow \quad \sup_\tau [u_{\tau,\beta}]_{C^{\alpha+\varepsilon'}} \leq C$$

$$\Rightarrow \quad \sup_\tau [u_{\tau,\beta+\alpha}]_{C^{\varepsilon''}} \leq C$$

Heuristically, we are borrowing a bit of the regulaity gained in the previous step to have compactness and therefore have control of the case when τ is small.

Recap

For

$$\begin{aligned} \partial_t u - Iu = f(x,t) \quad & \text{in} \quad Q_1 = B_1 \times (-1,0] \\ u = g \quad & \text{on} \quad (\mathbb{R}^n \setminus B_1) \times (-1,0] \end{aligned}$$

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Do the previous hypothesis imply $\partial_t u \in L^\infty(Q_{1/2})$?

Applications

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Dzięki!