Two terms asymptotic estimates for spectral heat content for relativistic stable processes in a real line

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Outline

1. Introduction
   - Heat content and spectral heat content
2. Main result
3. Further questions

Heat content of D in $\mathbb{R}^d$

Let $D \subset \mathbb{R}^d$. Consider the following heat equation without boundary condition

$$\begin{cases} \frac{\partial}{\partial t} u_D^{\mathrm{BM}} = \Delta u_D^{\mathrm{BM}}, \\ \lim_{t \to 0} u_D^{\mathrm{BM}}(t,x) = 1_D(x), \quad x \in \mathbb{R}^d \setminus \partial D \text{ (initial condition)}. \end{cases}$$

Then $u_D^{\mathrm{BM}}(t,x) = \int_D p^{\mathrm{BM}}(t,x,y)dy$, where
$p^{\mathrm{BM}}(t,x,y) = \frac{1}{(4\pi)^{d/2}} e^{-\frac{|x-y|^2}{4t}}$.
The heat content of D in $\mathbb{R}^d$ is defined by

$$H_D^{\mathrm{BM}}(t) = \int_D u_D^{\mathrm{BM}}(t,x)dx = \iint_{D \times D} p^{\mathrm{BM}}(t,x,y)dydx.$$

Spectral heat content of D 1/2

Now consider the following heat equation with Dirichlet boundary condition

$$\begin{cases} \frac{\partial}{\partial t} v_D^{\mathrm{BM}} = \Delta v_D^{\mathrm{BM}}, \\ \lim_{t\to 0} v_D^{\mathrm{BM}}(t,x) = 1_D(x), \quad x \in \mathbb{R}^d \setminus \partial D, \text{ (initial condition)} \\ v^{\mathrm{BM}}(t,x) = 0, \quad x \in \partial D, t > 0 \text{ (Dirichlet boundary condition).} \end{cases}$$

Then $v_D^{\mathrm{BM}}(t,x) = \mathbb{P}_x\left(\tau_D^{\mathrm{BM}} > t\right) = \int_D p_D^{\mathrm{BM}}(t,x,y)dy$,
$\tau_D^{\mathrm{BM}} = \inf\{s > 0 : W_s \notin D\}$.
The spectral heat content $Q_D^{\mathrm{BM}}(t)$ is defined by

$$Q_D^{\mathrm{BM}}(t) = \int_D v_D^{\mathrm{BM}}(t,x)dx = \int_D \mathbb{P}_x(\tau_D^{\mathrm{BM}} > t)dx.$$

Spectral heat content of D 2/2

When D has a finite volume, there exist eigenvalues and eigenfunctions $\{(\lambda_n^{\mathrm{BM}}, \phi_n^{\mathrm{BM}})\}_{n=1}^{\infty}$ with $0 < \lambda_1^{\mathrm{BM}} < \lambda_2^{\mathrm{BM}} \le \cdots \to \infty$ and $p_D^{\mathrm{BM}}(t,x,y) = \sum_{n=1}^{\infty} e^{-\lambda_n^{\mathrm{BM}} t} \phi_n^{\mathrm{BM}}(x)\phi_n^{\mathrm{BM}}(y)$.
Note that

$$
\begin{aligned}
Q_D^{\mathrm{BM}}(t) &= \int_D v_D^{\mathrm{BM}}(x)dx \\
&= \iint_{D\times D} p_D^{\mathrm{BM}}(t,x,y)dydx \\
&= \iint_{D\times D} \sum_{n=1}^{\infty} e^{-t\lambda_n^{\mathrm{BM}}} \phi_n^{\mathrm{BM}}(x)\phi_n^{\mathrm{BM}}(y)dydx \\
&= \sum_{n=1}^{\infty} e^{-t\lambda_n^{\mathrm{BM}}} \int_D \phi_n^{\mathrm{BM}}(x)dx \int_D \phi_n^{\mathrm{BM}}(y)dy \\
&= \sum_{n=1}^{\infty} e^{-t\lambda_n^{\mathrm{BM}}} \left(\int_D \phi_n^{\mathrm{BM}}(x)dx\right)^2 .
\end{aligned}
$$

Asymptotic expansion of spectral heat content for Laplacian

1. van den Berg and Davies, 1989
   Let D be bounded and connected with $C^{1,1}$ boundary ∂D.

$$\left| Q_D^{\mathrm{BM}}(t) - \left(|D| - \frac{2|\partial D|}{\sqrt{\pi}} t^{1/2} \right) \right| \leq 10^d |D| \frac{t}{R^2}.$$

2. van den Berg and Le Gall, 1994
   Let D be bounded and connected with C^3 boundary ∂D.

$$\left| Q_D^{\mathrm{BM}}(t) - \left(|D| - \frac{2|\partial D|}{\sqrt{\pi}} t^{1/2} + \frac{(d-1)\int_{\partial D} H(s)ds}{2} t \right) \right| \leq c t^{3/2}.$$

Asymptotic expansion of heat content for Laplacian

Let $D \subset \mathbb{R}^d$ be a bounded $C^{1,1}$ open set.

1. Miranda et all, 2007
$$H_D^{\mathrm{BM}}(t) = |D| - \frac{|\partial D|}{\sqrt{\pi}} t^{1/2} + o(t^{1/2}), \quad t \to 0.$$
2. van den Berg and Gittens, 2015
$$\left| H_D^{\mathrm{BM}}(t) - \left(|D| - \frac{|\partial D|}{\sqrt{\pi}} t^{1/2} \right) \right| \leq 2^{d+2} d^3 |D| \frac{t}{R^2}.$$

Asymptotic expansion of heat content and spectral heat content for fractional Laplacian

1. Valverde, preprints.
   Let $D = (a, b)$ be a bounded open interval in $\mathbb{R}^1$.

$$\lim_{t \downarrow 0} \frac{|D| - Q_D^{\mathrm{SSP}}(t)}{f_\alpha(t)} = \begin{cases} 2\mathbb{E}[\overline{X_1}], & 1 < \alpha \le 2, \\ \frac{2}{\pi}, & \alpha = 1, \\ A_{1,\alpha}\mathcal{P}_\alpha(D), & 0 < \alpha < 1, \end{cases}$$

where $f_\alpha(t) = \begin{cases} t^{1/\alpha} & \text{if } 1 < \alpha \le 2, \\ t\ln(\frac{1}{t}) & \text{if } \alpha = 1, \\ t & \text{if } 0 < \alpha < 1, \end{cases}$ $\quad \overline{X_1} = \sup\{X_s : 0 \le s \le 1\}$,

$A_{d,\alpha} = \alpha 2^{\alpha-1}\pi^{-1-\frac{d}{2}} \sin\left(\frac{\pi\alpha}{2}\right)\Gamma\left(\frac{d+\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)$, and
$\mathcal{P}_\alpha(D) = \int_D \int_{D^c} \frac{1}{|x-y|^{d+\alpha}} dxdy$.

Main question

What happens if we replace Brownian motions by some other Lévy processes different from stable processes?

What kind of information does non-local spectral heat content of the processes have?

In this talk, we will focus on spectral heat content of relativistic stable processes(RSP) in $(a, b) \subset \mathbb{R}^1$.

Relativistic stable processes

For any $m > 0$ and $\alpha \in (0, 2)$, relativistic α-stable processes X^m on $\mathbb{R}^d$ with mass m are Lévy processes with characteristic functions given by

$$\mathbb{E}\left[\exp(i\xi \cdot (X_t^m - X_0^m))\right] = \exp(-t((|\xi|^2 + m^{2/\alpha})^{\alpha/2} - m)), \quad \xi \in \mathbb{R}^d.$$

RSP behave like discontinuous stable processes in a small scale and behave like Brownian motions in a large scale.

Main result

Theorem (P. and Song, in preparation)

Let $1 \leq \alpha < 2$ and $D = (a, b)$ be a bounded open interval in $\mathbb{R}^1$. Then we have

$$\lim_{t \to 0} \frac{|D| - Q_D^m(t)}{f_\alpha(t)} = \lim_{t \to 0} \frac{|D| - Q_D^{SSP}(t)}{f_\alpha(t)}.$$

Hence for $1 \leq \alpha < 2$ we have

$$Q_D^m(t) = |D| - c_\alpha f_\alpha(t) + o(f_\alpha(t)) \quad \text{as } t \to 0.$$

Heat trace for RSP

Let D be a bounded $C^{1,1}$ open set in $\mathbb{R}^d$, $d \geq 2$ and
$Z_D^{\mathrm{SSP}}(t) = \int_D p_D^{\mathrm{SSP}}(t,x,x)dx = \sum_{n=1}^{\infty} e^{-\lambda_n^{\mathrm{SSP}} t}$ and
$Z_D^{\mathrm{RSP}}(t) = \int_D p_D^{\mathrm{RSP}}(t,x,x)dx = \sum_{n=1}^{\infty} e^{-\lambda_n^{\mathrm{RSP}} t}$ be trace of symmetric stable processes and relativistic stable processes, respectively.

Theorem (Bañuelos and Kulczycki 2008)

$$Z_D^{SSP}(t) = C_1|D|t^{-\frac{d}{\alpha}} - C_2|\partial D|t^{\frac{1-d}{\alpha}} + O(\frac{t^{2/\alpha}}{t^{d/\alpha}}), \quad t \to 0.$$

Theorem (P. and Song 2014)

As $t \to 0$

$$Z_D^{RSP}(t) = C_1|D|t^{-\frac{d}{\alpha}} - C_2|\partial D|t^{\frac{1-d}{\alpha}} + \frac{\omega_d \Gamma(d/\alpha)|D|}{(2\pi)^d \alpha} t^{-\frac{d}{\alpha}} \sum_{n=1}^{k} \frac{m^n}{n!} t^n + O(\frac{t^{2/\alpha}}{t^{d/\alpha}}).$$

$k = \sup\{n : n < \frac{2}{\alpha}\}$, $C_1 = \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^d \alpha}$, and $C_2 = C_2(d,\alpha)$.

Further questions

Short term goals

1. $0 < \alpha < 1$.
2. Higher dimensions $d \geq 2$.
3. Asymptotic spectral heat content expansion for general open sets in $\mathbb{R}^1$.
   $Q_\Omega(t)$, where $\Omega = \cup_{i=1}^{\infty}(a_i, b_i)$ with $\sum_{i=1}^{\infty}(b_i - a_i) < \infty$.

Long term goal

1. What kind of geometric information about the domain D can one obtain from non-local spectral heat content $Q_D(t)$?

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Thank you!