Bessel heat kernel estimates

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Probabilistic interpretation: $k(t,x,y)$ is the transition probability density for Brownian motion $W = (W(t))$ in $\mathbf{R}^d$.

We add some excitement by considering the heat equation on some (nice) subset $D \subset \mathbf{R}^d$ with Dirichlet boundary conditions

$$\begin{aligned} \Delta k_D(t,x,y) &= \frac{\partial}{\partial t} k_D(t,x,y), \quad x,y \in D, \quad t>0, \\ \lim_{t\to 0} k_D(t,x,y) &= \delta_x(y), \qquad x,y \in D, \\ k_D(t,x,y) &= 0, \qquad x \in \partial D \text{ or } y \in \partial D. \end{aligned}$$

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Notation: We write $f(t,x,y) \approx g(t,x,y)$ whenever there exists constant $C > 1$ such that

$$\frac{1}{C} f(t,x,y) \leq g(t,x,y) \leq C\, f(t,x,y)$$

for indicated range of t and space variables x and y.

Behaviour for large t

- The spectral representation

$$k_D(t,x,y) = \sum_{k=1}^{\infty} e^{-\lambda_k^D t} \varphi_k^D(x) \varphi_k^D(y)$$

where λ_k^D and φ_k^D are eigenvalues and eigenvectors of Δ on D, i.e.

$$\begin{aligned} \Delta \varphi_k^D(x) &= -\lambda_k^D \varphi_k^D(x), \quad x \in D, \\ \varphi_k^D(x) &= 0, \qquad x \in \partial D. \end{aligned}$$

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- For t large enough the first term dominates the others and we have

$$k_D(t,x,y) \approx e^{-\lambda_1^D t}\varphi_1^D(x)\varphi_1^D(y), \quad x, y \in D.$$

- For t small this representation seems to be useless.

Behaviour for small t

- In 1987 E.B. Davies showed that there exists constants $C, c > 0$ such that

$$k_D(t,x,y) \leq C\left(\frac{\delta_D(x)\delta_D(y)}{t} \wedge 1\right)\frac{1}{t^{d/2}}\exp\left(-c\frac{|x-y|^2}{t}\right)$$

  for every $x, y \in D$ and $t < T$.
    - D is a bounded $C^{1,1}$ domain
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  - D is a bounded $C^{1,1}$ domain
  - $\delta_D(x)$ stands for a distance of x to boundary ∂D.
- In 2002 Q.S. Zhang 2002 showed the lower bounds of the similar kind

$$k_D(t,x,y) \approx \left(\frac{\delta_D(x)\delta_D(y)}{t} \wedge 1 \right) \frac{1}{t^{d/2}} \exp\left(-c_i \frac{|x-y|^2}{t} \right)$$

  for $x, y \in D$ and $t < T$.

Question: Can we describe the exponential behaviour of

$$k_D(t,x,y),$$

i.e. get the same constants in the exponential terms in the lower and upper bounds? At least for a unit ball $B(0,1)$?

Related question:

- Let us consider the radial version of the problem.

$$\Delta f = L_\nu g$$

for a radial function $f(x) = g(|x|)$, where

$$L_\nu = \frac{d^2}{dx^2} + \frac{2\nu+1}{2}\frac{d}{dx}$$

is the Bessel operator of order $\nu = d/2 - 1$.

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- For $\nu > -1$ we define the Fourier-Bessel system

$$\varphi_n^{(\nu)}(x) = \frac{\sqrt{2}}{J_{\nu+1}(j_{n,\nu})} x^{-\nu} J_\nu(j_{n,\nu}x), \quad n = 1, 2, \ldots,$$

where $(j_{n,\nu})_n$ are positive zeros of Bessel functions $J_\nu(z)$.

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- $(\varphi_n^{(\nu)})_n$ - an orthonormal basis in $L^2((0,1), dm^{(\nu)})$, where

$$m^{(\nu)}(dx) = x^{2\nu+1}dx$$

and $\varphi_n^{(\nu)}$ are eigenfunctions of $-L_\nu$ with eigenvalue $j_{n,\nu}^2$

- The corresponding Fourier-Bessel heat kernel is given by

$$G_t^{(\nu)}(x,y) = 2(xy)^{-\nu}\sum_{n=1}^{\infty} \exp\left(-j_{n,\nu}^2 t\right) \frac{J_\nu(j_{n,\nu}x)J_\nu(j_{n,\nu}y)}{|J_{\nu+1}(j_{n,\nu})|^2}.$$

  where $x, y \in (0,1)$ and $t > 0$.
- $G_t^{(\nu)}(x,y)$ is the integral kernel of a semigroup $\exp(-t\mathcal{L}_\nu)$
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Probabilistic interpretation: We have

$$G_t^{(\nu)}(x,y) = p_{[0,1)}(2t, x, y),$$

where $p_{[0,1)}(t,x,y)$ is the transition probability density of the Bessel processes killed when leaving interval $[0,1)$.

Fourier-Bessel heat kernel

Theorem (JM, G. Serafin, T. Zorawik, JMAA 2015)

For $\nu > -1$, we have

$$G_t^{(\nu)}(x,y) \stackrel{\nu}{\approx} \frac{(1+t)^{\nu+2}}{(t+xy)^{\nu+1/2}} \left(1 \wedge \frac{(1-x)(1-y)}{t}\right) \frac{1}{\sqrt{t}} e^{-\frac{|x-y|^2}{4t} - j_{1,\nu}^2 t},$$

for every $x, y \in [0,1)$ and $t > 0$.

- Only small t is important.
- The exponential behaviour is the same as for the global kernel

$$p^{(\nu)}(2t,x,y) = \frac{(xy)^{-\nu}}{2t} \exp\left(-\frac{x^2+y^2}{4t}\right) I_\nu\left(\frac{xy}{2t}\right),$$

$$p^{(\nu)}(2t,0,y) = \frac{1}{2^{2\nu+1} t^{\nu+1} \Gamma(\nu+1)} \exp\left(-\frac{y^2}{4t}\right).$$

- The proof is based on the probabilistic approach.

We consider the corresponding problem on a half-line $(1, \infty)$.

- There is series representation since the set is unbounded and the operator has a continuous spectrum.
- The problem has sense for

$$L_\mu = \frac{d^2}{dx^2} + \frac{2\mu+1}{2}\frac{d}{dx}$$

  for every $\mu \in \mathbf{R}$.
- In this case the large times t are also (mainly!) problematic.

Probabilistic interpretation: we consider the transition probability density

$$p^{(\mu)}_{(1,\infty)}(t,x,y)$$

of the Bessel process killed upon leaving a half-line $(1, \infty)$.

Bessel heat kernel of a half-line $(1,\infty)$

Theorem (K. Bogus, JM, Potential Anal. 2014)

For $\mu \neq 0$ *we have*

$$p^{(\mu)}_{(1,\infty)}(t,x,y) \overset{\mu}{\approx} \left[1 \wedge \frac{(x-1)(y-1)}{t}\right] \left(1 \wedge \frac{xy}{t}\right)^{|\mu|-\frac{1}{2}} \frac{\exp\left(-\frac{(x-y)^2}{2t}\right)}{(xy)^{\mu+1/2}\sqrt{t}}$$

for every $x, y > 1$ *and* $t > 0$

- Once again the exponential behaviour is the same as for the global heat kernel.

Bessel heat kernel of a half-line $(1,\infty)$

Theorem (K. Bogus, JM, Math.Nachr. 2016)

We have

$$\frac{p^{(0)}_{(1,\infty)}(t,x,y)}{p^{(0)}(t,x,y)} \approx 1 \wedge \left[\ln(x)\ln(y)\left(\ln\frac{3xy+3t}{x+\sqrt{t}}\ln\frac{3xy+3t}{y+\sqrt{t}}\right)^{-1}\left(1+\frac{xy}{t}\right)\right]$$

for every $x, y > 1$ and $t > 0$.

Equivalently

$$p^{(0)}_{(1,\infty)}(t,x,y) \approx \ln x \ln y \left(\ln\frac{3t}{x+\sqrt{t}}\ln\frac{3t}{y+\sqrt{t}}\right)^{-1}\frac{1}{t}\exp\left(-\frac{x^2+y^2}{2t}\right)$$

for $xy \le t$ and

$$p^{(0)}_{(1,\infty)}(t,x,y) \approx \left(1 \wedge \frac{(x-1)(y-1)}{t}\right)\frac{1}{\sqrt{xyt}}\exp\left(-\frac{(x-y)^2}{2t}\right)$$

for $xy \ge t$.

What about our original problem for a ball $B(0,1)$ and Laplacian?

Recall

- We consider the Dirichlet heat kernel

$$k_{B(0,1)}(t,x,y)$$

  of a centred unit ball $B(0,1)$
- the Zhang's result

$$k_{B(0,1)}(t,x,y) \approx \left(\frac{(1-|x|)(1-|y|)}{t} \wedge 1\right) \frac{1}{t^{d/2}} \exp\left(-c_i \frac{|x-y|^2}{2t}\right)$$

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  for t small enough.
- Is the exponential behaviour the same as for the Gaussian kernel, i.e. $c_1 = c_2$?

Classical Dirichlet heat kernel of a ball

Theorem (JM, Serafin, 2016)

We have

$$k_{B(0,1)}(t,x,y) \overset{d}{\approx} h(t,x,y)\frac{1}{t^{d/2}}\exp\left(-\frac{|x-y|^2}{2t}\right)$$

for every $x, y \in B(0,1)$ and t small enough. Here $h(t,x,y)$ is equal to

$$\left(\frac{(1-|x|)(1-|y|)}{t}\wedge 1\right)+\left[\frac{(1-|x|)|x-y|^2}{t}\wedge 1\right]\left[\frac{(1-|y|)|x-y|^2}{t}\wedge 1\right]$$

- The Zhang's result

$$k_{B(0,1)}(t,x,y)\approx\left(\frac{(1-|x|)(1-|y|)}{t}\wedge 1\right)\frac{1}{t^{d/2}}\exp\left(-c_i\frac{|x-y|^2}{2t}\right)$$

The starting point for all of those results is the Hunt formula

$$k_D(t,x,y) \;=\; k(t,x,y) - \mathbf{E}_x[t > T_D; W(t-T_D) \in dy]$$

where

- T_D is the first exit time of W from D.
- Consequently, we have

$$k_D(t,x,y) \;=\; k(t,x,y) - \int_0^t \int_{\partial D} k(t-s,x,z) q_x(ds,dz)$$

where $q_s(ds,dz)$ is a joint distribution of the first hitting time T_D and the hitting place $W(T_D)$.

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Better than the highly oscillating series representation.

Thank you very much.