Universität Bielefeld

Regularity of solutions to anisotropic nonlocal equations

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Anisotropic Lévy process

Assume $d \in \mathbb{N}$, $d \geq 2$.
Let $(L_t^i)_{t\geq 0}$, $i = 1, \ldots, d$ be independent one-dimensional α_i stable Lévy processes with $\alpha_i \in (0, 2)$.

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Then L is a pure jump Lévy process with Lévy measure

$$\nu(dw) = \sum_{k=1}^{d} \left(\frac{c(\alpha_k)}{|w_k|^{1+\alpha_k}} dw_k \left(\prod_{\substack{j=1 \\ j\neq k}} \delta_{\{0\}}(dw_j) \right) \right).$$

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Hence the generator is given by

$$\mathcal{L}f(x) = \sum_{k=1}^{d} \int_{\mathbb{R}\setminus\{0\}} (f(x+he_k) - f(x) - \mathbb{1}_{\{|h|\leq 1\}} \partial_k f(x) h) \frac{c(\alpha_k)}{|h|^{1+\alpha_k}} dh.$$

Anisotropic Lévy process

Generator $\mathcal{L}$ of the anisotropic stable process:

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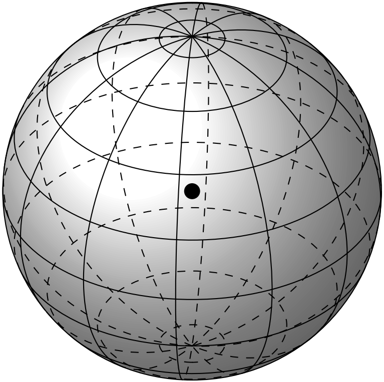
Generator $\mathcal{L}$ of the isotropic (rotationally symmetric) stable process:

$$-(-\Delta)^{\alpha/2} = \int_{\mathbb{R}^d\setminus\{0\}} (f(x+w) - f(x) - \mathbb{1}_{\{|w|\leq 1\}} \nabla f(x)\cdot w) \frac{c(d,\alpha)}{|w|^{d+\alpha}} dw.$$

Anisotropic Lévy process

Isotropic case: $\nu(dw) = \dfrac{c(d,\alpha)}{|w|^{d+\alpha}} dw.$

Supported on whole $\mathbb{R}^d$.



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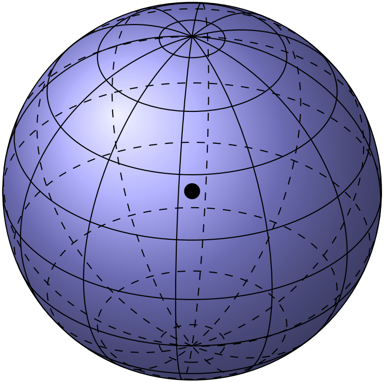

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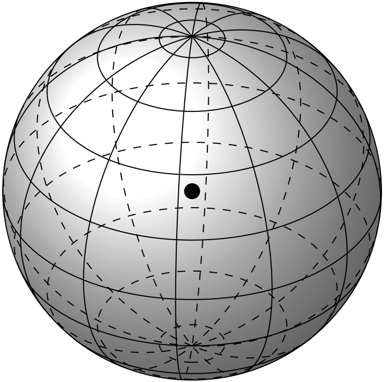

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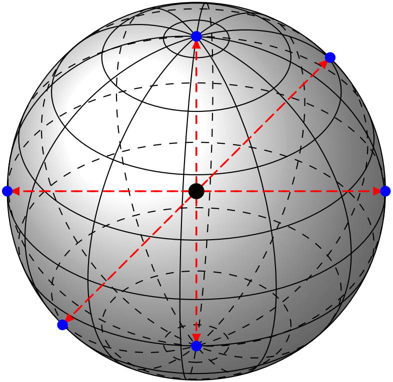

We consider the system of SDEs

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If there exists a unique solution to the system (1), which is a Markov process, then the infinitesimal generator is given by (c.f. [BC06] in the case $\alpha_i = \alpha$ for all i)

$$\mathcal{L}f(x) = \sum_{j=1}^{d} \int_{\mathbb{R}\setminus\{0\}} (f(x + a_j(x)h) - f(x) - h\mathbb{1}_{\{|h|\leq 1\}}\nabla f(x) \cdot a_j(x)) \frac{c_{\alpha_j}}{|h|^{1+\alpha_j}} dh,$$

where $a_j(x)$ denotes the jth column of the matrix $(A_{ij}(x))$ and $f \in C_b^2(\mathbb{R}^d)$.

The martingale problem

Definition

Let $\mathcal{L}$ be an operator whose domain includes $C_b^2(\mathbb{R}^d)$. A probability measure $\mathbb{P}^{x_0}$ on the space $\mathcal{D}([0,\infty))$ is a solution to the martingale problem for $\mathcal{L}$ started at x_0 if $X_t(\omega) = \omega(t)$ are the coordinate maps, $\mathcal{F}_t$ is the σ-field generated by the cylindrical sets and the following two conditions hold:

1. $\mathbb{P}^{x_0}(X_0 = x_0) = 1$.
2. For each $f \in C_b^2$

$$f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)ds$$

is a $\mathbb{P}^{x_0}$-martingale.

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1. Existence (resp. uniqueness) of a weak solution to (1) is equivalent to existence (resp. uniqueness) of a solution to the martingale problem for $\mathcal{L}$ with starting point x_0.

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2. Uniqueness of a solution to the martingale problem implies the strong Markov property of $(X, \mathbb{P}^{x_0}, x_0 \in \mathbb{R}^d)$.
3. Existence of a solution $\mathbb{P}^x$ to the martingale problem for $\mathcal{L}$ implies the existence of a solution to the Cauchy problem for the operator $\mathcal{L}$ with initial data in C^2. For $f \in C^2$

$$u(x,t) = \mathbb{E}^x(f(X_t))$$

is a solution to

$$\begin{cases} \partial_t u = \mathcal{L}u & \text{in } \mathbb{R}^d \times (0,\infty), \\ u_0 = f & \text{for } t = 0. \end{cases}$$

The martingale problem

$$dX_t^i = \sum_{j=1}^{d} A_{ij}(X_{t-})dL_t^j, \quad i = 1, \ldots, d$$
$$X_0 = x_0.$$

3. We assume that there is a unique weak solution to the system (1) and the family $\{X, \mathbb{P}^{x_0}, x_0 \in \mathbb{R}^d\}$ forms a strong Markov process on $\mathbb{R}^d$.

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3. We assume there is a unique weak solution to the system (1) and the family $\{X, \mathbb{P}^x, x \in \mathbb{R}^d\}$ forms a strong Markov process on $\mathbb{R}^d$.

In the case $\alpha_i = \alpha$ for all $i \in \{1, \dots, d\}$ the existence and weak uniqueness to (1) was proved by [BC06]

Aim of this talk

We will show that harmonic functions, associated to solutions of the system of SDEs, fulfill a Hölder estimate.

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Theorem

Let $r \in (0,1]$ and $x_0 \in \mathbb{R}^d$. Suppose h is harmonic in $M_r^2(x_0)$ with respect to X and h is bounded in $\mathbb{R}^d$. There exists positive constants c_1 and β, that depend on Λ and the modulus of continuity of A, but otherwise is independent of h and r, such that

$$|h(x) - h(y)| \leq c_1 \left(\frac{|x-y|}{r^{\gamma/\min\{\alpha_1,\dots,\alpha_d\}}} \right)^{\beta} \sup_{\mathbb{R}^d} |h(z)| \quad \textit{for } x,y \in M_r^1(x_0),$$

where γ is the geometric mean of $\alpha_1, \dots, \alpha_d$.

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Basic ideas of the proof!

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Basic ideas of the proof!
The case $\alpha_i = \alpha$ for all i is covered in [BC10].

Let $T_D = \inf\{t > 0 : X_t \in D\}$ be the first entrance time of X and $\tau_D : \inf\{t > 0\colon X_t \notin D\}$ be the first exit time of X.

Definition

A bounded function $h : \mathbb{R}^d \to \mathbb{R}$ is called harmonic with respect to a process X in a domain D, if $h(X_{t\wedge\tau_D})$ is a martingale with respect to $\mathbb{P}^{x_0}$.

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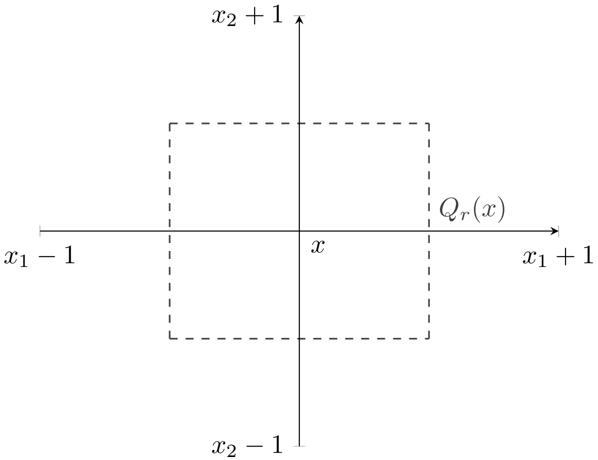
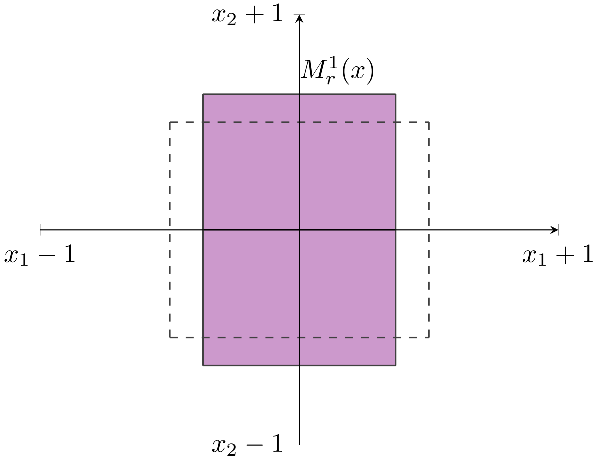
$$M_r^k(x) = \bigtimes_{i=1}^{d} \left(x_i - kr^{\gamma/\alpha_i}, x_i + kr^{\gamma/\alpha_i} \right).$$

Cube with radius $r = 1/2$ and center x



Given $\alpha_1 = 0.7$ and $\alpha_2 = 1.5$. The set $M^1_{1/2}(x)$



Lemma

Let $x \in \mathbb{R}^d$ *and* $r \in (0,1]$. *Then there exists a constant* $c_1 > 0$, *depending on* Λ, *such that*

$$\mathbb{E}^z\left[\tau_{M_r(x)}\right] \leq c_1 r^\gamma \quad \textit{for all } z \in M_r(x).$$

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Lemma

Let $x \in \mathbb{R}^d, r \in (0,1]$ *and* $R \geq 2r$. *There exists a constant* $c_1 > 0$*, depending on* Λ*, such that*

$$\mathbb{P}^z(X_{\tau_{M_r(x)}} \notin M_R(x)) \leq c_1 \left(\frac{r}{R}\right)^\gamma \quad \text{for all } z \in M_r(x).$$

Main ingredient for the proof

Theorem

Let $r \in (0,1]$, $x_0 \in \mathbb{R}^d$ *and* $M := M_r^1(x_0)$. *There exists a nondecreasing function* $\varphi : (0,1) \to (0,1)$ *such that*

$$\mathbb{P}^{x_0}(T_A < \tau_M) \geq \varphi(|A|)$$

for all $x \in M_r^{1/2}(x_0)$ *and* $A \subset M$ *with* $|A| > 0$.

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For the SDE

$$dX_t = \sigma(X_t)dW_t + b(X_t)dt, \quad X_0 = x_0$$

a proof of this theorem can be found in the book [Bas98].

Support Theorem

Gives information about the topological support of $\mathbb{P}^{x_0}$.

Theorem

Let $r \in (0,1]$. *Let* $x_0 \in \mathbb{R}^d$, $\varepsilon \in (0, r^{\gamma/\min\{\alpha_1,\ldots,\alpha_d\}})$ *and* $t_0 > 0$. *Let* $\varphi : [0,t_0] \to \mathbb{R}^d$ *be continuous with* $\varphi(0) = x_0$ *and* $\varphi([0,t_0]) \subset M_r^1(x_0)$. *There exists* $c_1 > 0$ *depending on* Λ, *the modulus of continuity of* $A(x)$, φ, ε *and* t_0 *such that*

$$\mathbb{P}^{x_0}\left(\sup_{s \le t_0} |X_s - \varphi(s)| < \varepsilon\right) > c_1.$$

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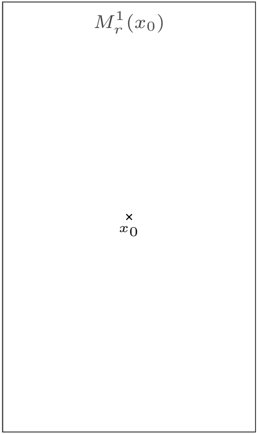
In [BC10] Richard Bass and Zhen-Qing Chen considered the system (1) with $\alpha_i = \alpha$ for all $i \in \{1,\dots,d\}$ and gave a proof of the support theorem in this case.

Support Theorem



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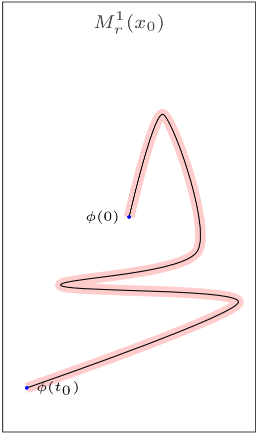

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