Porous media equation in tubular domains: large time behaviour of solutions

Jan Goncerzewicz

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Joint work with:
Brian H. Gilding, Kuwait University

The Problem

We consider the homogeneous Cauchy–Dirichlet problem

$$\begin{cases} \partial_t u = \Delta u^m & \text{for } (x,t) \in \Omega \times \mathbb{R}_+ \\ u(x,t) = 0 & \text{for } (x,t) \in \partial\Omega \times \mathbb{R}_+ \\ u(x,0) = u_0(x) & \text{for } x \in \Omega, \end{cases} \tag{1}$$

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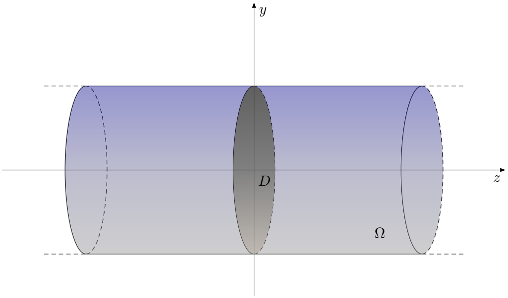
in which

- $m > 1$,
- $u_0 \in L^1_{\mathrm{loc}}(\Omega)$, $u_0 \geq 0$ (a.e.),
- Ω is a unbounded connected open subset of $\mathbb{R}^N$ of specific shape.

The Ω: an infinite cylinder case

- $\Omega = D \times \mathbb{R}$, $D \subset \mathbb{R}^n$, $(N = n + 1)$, where
  - D is open, bounded, connected,
  - ∂D is locally Lipschitz continuous and satisfies a uniform interior ball condition.



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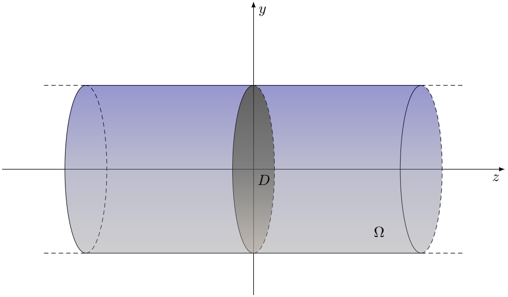

Notation:
$x \in \Omega$, $x = (y, z)$,
with $y \in D$, $z \in \mathbb{R}$

Definition 1

A weak solution of equation

$$\partial_t u = \Delta u^m \quad \text{for } (x,t) \in Q \subset \Omega \times \mathbb{R}_+ \tag{2}$$

is a nonnegative function $u \in L^1_{\mathrm{loc}}(Q)$ for which every component of ∇u^m exists as a weak derivative in $L^1_{\mathrm{loc}}(Q)$, and

$$\iint_Q \{(\nabla u^m \cdot \nabla \psi - u \partial_t \psi\} \, dx \, dt = 0 \quad \text{for all } \psi \in C_0^1(Q).$$

A strong solution is a weak solution such that $\partial_t u \in L^1_{\mathrm{loc}}(Q)$.

Preliminaries: main assumptions and definitions cont.

Definition 2

A solution of problem (1) is a strong solution u of (2) in $Q = \Omega \times \mathbb{R}_+$ such that

- trace of $u^m(\cdot, t)$ on $\partial\Omega$ is defined and equal to zero for almost all $t \in \mathbb{R}_+$,
- $u(\cdot, t) \to u_0$ as $t \downarrow 0$ in $L^1_{\mathrm{loc}}(\overline{\Omega})$.

Preliminaries: existence, uniqueness, regularity

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- $u_0 \in L^\infty(\Omega) \Rightarrow u \in L^\infty(\Omega \times \mathbb{R}_+)$,
- $u_0 \in C(\Omega) \Rightarrow u \in C(\Omega \times [0, \infty))$,

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- $u_0 \in C(\Omega) \Rightarrow u \in C(\Omega \times [0, \infty))$,
- $u_0 \in C(\overline{\Omega})$ and $u_0 = 0$ on $\partial\Omega \Rightarrow u \in C(\overline{\Omega} \times [0, \infty))$.

Preliminaries: positivity set

- if u_0 is nontrivial and has bounded support, then the positivity set

$$P(t) = \{x \in \Omega : u(x,t) > 0\}$$

is bounded for all $t > 0$ and connected for large enough t,

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- for every $y \in D$ there is a $T \geq 0$ such that

$$\{z \in \mathbb{R} : x = (y,z) \in P(t)\} = (\gamma^-(t,y), \gamma^+(t,y)) \quad \text{for all } t > T$$

for some functions $\gamma^\pm(\cdot, y) : (T, \infty) \to \mathbb{R}_+$.

Subject of investigations

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- Gilding & G. results (Interfaces Free Bound. 2016)

Part I

Vázquez results (Comm.Contemporary Math 2007)

Self-similar solution I

- The Friendly Giant, (Aronson & Peletier 1981, Dahlberg & Kenig 1988, Vázquez 2004)

$$\widetilde{U}(y,t) = t^{-\mu} F(y), \quad (y,t) \in \overline{D} \times \mathbb{R}_+ \,,$$

where $\mu = 1/(m-1)$ and

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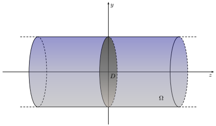

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- $F > 0$ in D $\quad (\Rightarrow F \in C(\overline{D}) \cap C^\infty(D)$ is a classical solution of this problem).

Self-similar solution II (J.L. Vázquez 2007)

- Travelling wave solutions along z variable

$$U(x,t;a) = t^{-\mu} f(y, z - c\ln t - a), \quad (x,t) \in \Omega \times \mathbb{R}_+, \quad (a \in \mathbb{R})$$

for some $c = c(m, D) > 0$, where $\mu = 1/(m-1)$ and

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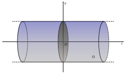

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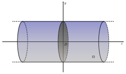

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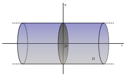

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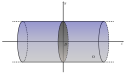

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Note:

- uniqueness of f not proven.
- $c(m, D)$ not given explicitly.

Vázquez's asymptotics results

- if $u_0 \in L^1(\Omega)$ then

$$t^{\mu}\,|u(x,t) - F(y)| \to 0 \text{ as } t \to \infty \qquad (\text{ recall } x = (y,z))$$

uniformly with respect to x in compact subsets of $\overline{\Omega}$,

- if $u_0 \in L^1(\Omega)$ has bounded support then

$$\gamma^{\pm}(t;\cdot) - c \ln t = \mathrm{o}(\ln t) \quad \text{as } t \to \infty$$

pointwise in D,

- if u_0 vanishes in some set $D \times (\ell, \infty)$ and u_0 satisfies a certain growth criterion as $y \to -\infty$ then

$$\gamma^{+}(t;\cdot) - c \ln t = \mathrm{O}(1) \quad \text{as } t \to \infty$$

uniformly in D.

Part II

Gilding & G. results (Interfaces Free Bound. 2016)

GG results (Gilding, G. 2016)

Part I: concerning f

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Part I: concerning f

- $$c(m, D) = \frac{1}{(m-1)\sqrt{\lambda}} \qquad (= \mu/\sqrt{\lambda}),$$

  where λ is the first eigenvalue for the Laplacian with homogeneous boundary conditions in D.
- f is unique modulo translation.

GG results cont.

Part II: convergence of the solution

- there is a number a for which

$$u(x,t) - U(x,t;a) \to 0 \quad \text{as } t \to \infty$$

uniformly with respect to $x \in D \times \mathbb{R}_+$ in an appropriate reference frame .

- a is determined explicitly from the initial-data function u_0.

GG results cont.

Part III: right-hand interface of $u(x, t)$

If support of u_0 contained in $D \times (-\infty, \ell)$ then:

GG results cont.

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If support of u_0 contained in $D \times (-\infty, \ell)$ then:

- $$\liminf_{t\to\infty} \gamma^+(t;y) - c\ln t \geq \sigma(y) + a$$

  at every $y \in D$.

GG results cont.

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- $$\limsup_{t\to\infty} \gamma^+(t;\cdot) - c\ln t \leq \bar{\sigma}(\cdot) + a,$$
  where $\bar{\sigma}$ denotes the concave envelope of σ, uniformly in D.

GG results cont.

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recall Vázquez:

$$\gamma^+(t;\cdot) - c\ln t = \mathrm{O}(1) \quad \text{as } t \to \infty$$

uniformly in D.

GG results cont.

Part IV: general domains

- the results extend to more general domains Ω such that

$$D \times \mathbb{R}_+ \subseteq \Omega \subseteq D \times \mathbb{R}.$$

GG results cont.

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Part V: higher dimensions

- some results extend to domains

$$D \times \{z \in \mathbb{R}^k : |z| > \varrho\} \subseteq \Omega \subseteq D \times \mathbb{R}^k \quad \text{for some } \varrho > 0,$$

where D is as before and $k = N - n \geq 2$. Especially, in this case

$$t^{1/(m-1)} u(x,t) \to f(y, r - c \ln t + \omega \ln |\ln t| - a) \text{ as } t \to \infty,$$

where $x = (y, z_1, \dots, z_k)$, $r = (z_1^2 + z_2^2 + \cdots + z_k^2)^{1/2}$, $\omega = \frac{k-1}{2\sqrt{\lambda}}$ and f, c are as before.

Main idea and some details

‘Conservation laws drive solutions

into equilibrium states’

Invariance principle for $\Omega = D \times \mathbb{R}$

- Let $G(x)$ be a positive classical solution of the problem

$$\begin{cases} \Delta G = 0 & \text{in } D \times \mathbb{R} \\ G = 0 & \text{on } \partial D \times \mathbb{R} \\ G(x) = \mathrm{o}(h(x)) & \text{as } z \to \infty \text{ uniformly w.r.t.} y \in D\,, \end{cases}$$

where $h(x) = \begin{cases} \ln|x| & \text{if } N = 2 \\ 1 & \text{if } N \geq 3. \end{cases}$

Theorem (Invariance Principle for $\Omega = D \times \mathbb{R}$)

Let $u_0 \in L^1_{\mathrm{loc}}(\overline{\Omega})$ be nonnegative. Then the solution u of problem (1) *satisfies*

$$\int_\Omega G(x)u(x,t)\,dx = \int_\Omega G(x)u_0(x)\,dx \qquad \textit{for all } t > 0.$$

Convergence of solution I, $(\Omega = D \times \mathbb{R}_+)$

- Change of variables $(x, t, u) \to (\xi, \tau, \hat{u})$, (notation: $\xi = (\eta, \zeta)$)

$$\eta = y,$$
$$\zeta = z - c \ln t,$$
$$\tau = \ln t,$$
$$\hat{u} = t^{\mu} u.$$

comes from:

$U(x, t; a) = t^{-\mu} f(y, z - c \ln t - a)$.

We have

- $(x, t) \in \Omega \Leftrightarrow (\xi, \tau) \in D \times \mathbb{R}^2$
- $\hat{u}(\xi, \tau) = e^{\mu\tau} u(\eta, \zeta + c\tau, e^{\tau})$
- $\hat{u}$ is a weak solution of

$$\partial_\tau \hat{u} = \Delta \hat{u}^m + c \partial_\zeta \hat{u} + \mu \hat{u} \tag{3}$$

in $D \times \mathbb{R}^2$.

- $\hat{U}(\xi, \tau; a) = U(\xi, 1; a) \quad \text{in } D \times \mathbb{R}^2$ stationary solution of (3)

Question

How to connect $u(x, t)$ with proper $U(x, t; a)$?

Invariance principle gives the link

- We have

$$G(x) = Y(y)e^{\sqrt{\lambda}z} \quad \text{for } x \in \overline{D} \times \mathbb{R},$$

where

- $\lambda > 0$ is the first eigenvalue of the eigenvalue problem for Laplacian with homogeneous Dirichlet boundary conditions in D,
- $Y \in C(\overline{D}) \cap C^\infty(D)$ is the corresponding eigenfunction:

$$\begin{cases} -\Delta Y = \lambda Y & \text{in } D \\ Y = 0 & \text{on } \partial D. \end{cases}$$

- Y is unique modulo multiplication with a constant. We normalize this constant, by assuming that $\int_D Y(y)\,dy = 1$. In this case $Y > 0$ in D.

The link cont.

For $U(x,t;a) = t^{-\mu} f(y, z - c \ln t - a)$ there holds:

- for any $a \in \mathbb{R}$ and for all $t > 0$:

$$\int_{D\times\mathbb{R}} G(x)U(x,t;a)\,dx = \kappa \cdot e^{\sqrt{\lambda}\,a}\, t^{\sqrt{\lambda}\,c-\mu}$$

$$\text{with } \kappa = \int_{D\times\mathbb{R}} Y(y)e^{\sqrt{\lambda}\,z} f(x)\,dx > 0.$$

- invariance principle gives

$$\sqrt{\lambda}\,c - \mu = 0 \quad \Rightarrow \quad c = \mu/\sqrt{\lambda}\,.$$

Link done

- As a consequence

$$\int_{D\times\mathbb{R}} G(x)U(x,t;a)\,dx = \kappa \cdot e^{\sqrt{\lambda}\,a}$$
$$\text{with } \kappa = \int_{D\times\mathbb{R}} Y(y)e^{\sqrt{\lambda}\,z} f(x)\,dx > 0.$$

- Thus if $\int_\Omega G(x)u_0(x)\,dx < \infty$ and $a = \frac{1}{\sqrt{\lambda}} \ln\left(\frac{1}{\kappa}\int_\Omega G(x)u_0(x)\,dx\right)$ then

$$\int_\Omega G(x)u(x,t)\,dx = \int_\Omega G(x)U(x,t;a)\,dx \quad \text{for all } t > 0.$$

Convergence of solution II ($\Omega = D \times \mathbb{R}_+$)

- Step 1. For $s > 0$ we consider shifted trajectories

$$\hat{u}_s(\xi,\tau) = \hat{u}(\xi, s+\tau) \quad \text{for } (\xi,\tau) \in Q_s,$$

where

$$Q_s = \{(\xi,\tau) \in D \times \mathbb{R}^2 : \zeta + c(s+\tau) > 0\}.$$

We have

- $\hat{u}_s(\xi,\tau) \le F(\eta)$ for all $(\xi,\tau) \in Q_s$ and $s > 0$.
- the family $\{\hat{u}_s\}_{s\ge\varsigma}$ is equicontinuous in $\overline{Q_\varsigma}$ for all $\varsigma > 0$ by DiBenedetto estimates (DiBenedetto 1983).

Therefore

- for some sequence $\{s_i\}$, $\hat{u}_{s_i}$ converges to a nonnegative function $\hat{u}_\infty$ as $i \to \infty$ uniformly on compact subsets of $\overline{D} \times \mathbb{R}^2$.

Convergence of solution II ($\Omega = D \times \mathbb{R}_+$)

- Step 2. We analyse a Lyapunov function

$$\mathcal{I}(\tau) = \int_{D\times\mathbb{R}} G(\xi)\left|\hat{u}_\infty(\xi,\tau) - U(\xi,1;a)\right| d\xi \quad \text{for } \tau \in \mathbb{R}.$$

and show that

$$\mathcal{I}(\tau) = 0 \text{ for } \tau \in \mathbb{R}.$$

From this we deduce that

- $\hat{u}_\infty = U(\cdot,1;a)$ in $D \times \mathbb{R}^2$,
- the whole family $\{\hat{u}_s\}_{s>0}$ converges to $U(\cdot,1;a)$ as $s \to \infty$ uniformly on compact subsets of $D \times \mathbb{R}^2$,
- $\hat{u}(\cdot,\tau)$ converges to $U(\cdot,1;a)$ as $\tau \to \infty$ uniformly on all compact subsets of $\overline{D} \times \mathbb{R}$. Moreover, if Hypotheseis 2 below holds then the convergence is uniform on all sets of the form $\overline{D} \times (\alpha,\infty)$.

Convergence results ($\Omega = D \times \mathbb{R}_+$)

Hypothesis 1

$u_0 \in L^1_{\mathrm{loc}}(\overline{\Omega})$ is nonnegative, positive on a subset of Ω of positive measure, and such that $Gu_0 \in L^1(\Omega)$.

Convergence results ($\Omega = D \times \mathbb{R}_+$)

Hypothesis 1

$u_0 \in L^1_{\rm loc}(\overline{\Omega})$ is nonnegative, positive on a subset of Ω of positive measure, and such that $Gu_0 \in L^1(\Omega)$.

Hypothesis 2

u_0 satisfies

$$\limsup_{R\to\infty} R^{-2\mu-N} \int_{\{x\in\Omega:|x|<R\}} u_0(x)\,dx < \infty$$

and is such that $u_0 = 0$ in $\{x \in \Omega : z > \ell\}$ for some $\ell \in \mathbb{R}$.

Convergence results ($\Omega = D \times \mathbb{R}_+$) cont.

Theorem

Assume that u_0 satisfies Hypothesis 1. Define a as

$$a = \frac{1}{\sqrt{\lambda}} \ln\left(\frac{1}{\kappa} \int_\Omega G(x) u_0(x)\, dx \right) \quad \left(\text{recall } \kappa = \int_{D\times\mathbb{R}} Y(y) e^{\sqrt{\lambda} z} f(x)\, dx \right).$$

Then given any $\alpha < \beta$ and $1 \le p \le \infty$, the solution u of problem (1) is such that

$$t^\mu \| u(\cdot, t) - U(\cdot, t; a) \|_{L^p(D\times(\alpha + c\ln t, \beta + c \ln t))} \to 0 \quad \text{as } t \to \infty.$$

Moreover, if Hypothesis 2 holds, then

$$t^\mu \| u(\cdot, t) - U(\cdot, t; a) \|_{L^p(D\times(\alpha + c\ln t, \infty)) \to 0 \quad \text{as } t\to\infty}.$$

Convergence results, $(\Omega = D \times \mathbb{R}_+)$ cont.

Theorem cont.

If in addition

$$\liminf_{z\to-\infty} u_0(x)/F(y) > 0 \quad (F(y) \text{ as in Giant })$$

uniformly with respect to $y \in D$, then

$$t^\mu \|u(\cdot,t) - U(\cdot,t;a)\|_{L^\infty(D\times(-\infty,\beta+c\ln t))} \to 0 \quad \text{as } t\to\infty$$

for all $\beta \in \mathbb{R}$, and hence when Hypothesis 2 holds,

$$t^\mu \|u(\cdot,t) - U(\cdot,t;a)\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t\to\infty.$$