Boundary Harnack principle for nonsymmetric stable-like operators on $C^{1,1}$-open sets

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Motivating example

Let $D := \{x \in \mathbb{R}^d : x_d > 0\}$ be the upper half space. The function

$$h(x) := \begin{cases} (x_d)^{\alpha/2}, & \text{if } x_d > 0, \\ 0, & \text{if } x_d \le 0 \end{cases}$$

is harmonic on D for the fractional Laplacian $\Delta^{\frac{\alpha}{2}}$, $\alpha \in (1,2)$, and vanishes on D^c. The boundary decay rate is $h(x) \sim \text{dist}(x, \partial D)^{\alpha/2}$.

Aim: Show that harmonic functions for α-stable like operators have boundary decay rate $\delta(x)^{\alpha/2}$, where $\delta(x) = \text{dist}(x, \partial D)$, on any $C^{1,1}$-open set D.

Definition

A Borel measurable function u on $\mathbb{R}^d$ is *harmonic* on an open subset $D \subset \mathbb{R}^d$ if $\mathbb{E}_x(|u(X_{\tau_B})|) < \infty$ and

$$u(x) = \mathbb{E}_x u(X_{\tau_B}), \quad \forall x \in B,$$

for any open set B relatively compact in D.

Background

Boundary Harnack principle (BHP):
Let D domain, $Q \in \partial D$. For any u, v positive harmonic functions on $D \cap B(Q, r)$, vanishing on $D^c \cap B(Q, r)$,

$$\frac{u(x)}{v(x)} \leq C \frac{u(y)}{v(y)}, \quad \text{for all } x, y \in D \cap B(Q, cr).$$

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First studied for Laplacian and local operators:

- Lipschitz domains: Ancona (1978), Wu (1978)
- BHP characterizes uniform domains: Aikawa (2001, 2004),
- uniform domains in Harnack-type Dirichlet spaces: Gyrya, Saloff-Coste (2011)
- uniform domains in fractals: L. (2015)
- weaker formulations of BHP: e.g. Ancona ('07); Bass, Burdzy ('91)

Non-local operators: next slide.

Boundary Harnack principle for non-local operators

Rotationally invariant stable processes:
Bogdan (1997): on bounded Lipschitz domains,
Song, Wu (1999): on bounded k-fat open sets,
Bogdan, Kulczycki and Kwasnicki (2008): on arbitrary open sets

Symmetric stable processes:
Bogdan, Stos, Sztonyk (2002) and Sztonyk (2003).

rotationally invariant Lévy process (on arbitrary open sets):
Kim, Song, Vondraček (2012).

fractional Laplacian on open sets in fractal spaces: Stos (2006)

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Bogdan, Kumagai and Kwasnicki (2015): BHP holds on arbitrary open sets in metric measure spaces, for many non-local operators. *Includes non-symmetric operators, but requires a dual process.*

BHP with explicit decay rates on $C^{1,1}$-open sets:

symmetric stable processes: Kulczycki (1997), Chen, Song (1998)

censored stable processes: Bogdan, Burdzy, Chen (2003)

killed subordinate Brownian motion: Kim, Song, Vondraček (2012)

subordinate Brownian motions: Kim, Song, Vondraček (2014), Kim, Mimica (2014)

perturbed operators and sums of independent operators: e.g., by Chen, Kim, Song; Chen et al.

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censored stable-like processes: Guan (preprint 2007)

Let $\alpha \in (1,2)$.
α-stable like operator

$$\Delta^{\frac{\alpha}{2},\kappa} u(x) := \mathcal{A}(d,-\alpha) \lim_{\epsilon \to 0} \int_{\{y \in \mathbb{R}^d : |x-y| > \epsilon\}} \frac{\kappa(x,y)(u(y)-u(x))}{|x-y|^{d+\alpha}} dy,$$

for $x \in \mathbb{R}^d$,
where $\mathcal{A}(d,-\alpha) := |\alpha| 2^{\alpha-1} \Gamma((d+\alpha)/2) \pi^{-n/2} / \Gamma(1-\alpha/2)$.

Case $\kappa \equiv 1$: Fractional Laplacian.

In general, κ may be non-symmetric.

In general, we assume that κ has the following properties:

1. (Uniform boundedness) There exist constants $C_1, C_2 \in (0, \infty)$ such that

$$C_1 \le \kappa(x, y) \le C_2, \quad \forall x, y \in \mathbb{R}^d.$$

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2. $\kappa(x, x+y) = \kappa(x, x-y)$ for all $x, y \in \mathbb{R}^d$.
3. There exist constants $C_3 \in (0, \infty)$ and $\rho \in (\alpha/2, \alpha)$ such that

$$|\kappa(x, y) - \kappa(x, x)| \leq C_3 |x - y|^\rho, \quad \forall x, y \in \mathbb{R}^d.$$

4. There exist constants $C_4 \in (0, \infty)$ and $\gamma \in (0, 1)$ such that

$$|\kappa(x, x+z) - \kappa(y, y+z)| \leq C_4 |x - y|^\gamma, \quad \forall x, y \in \mathbb{R}^d.$$

Definition

An open set $D \subset \mathbb{R}^d$ is called a $C^{1,1}$-open set with characteristics (R_0, Λ_0) if, for each boundary point $z \in \partial D$,
there is a C^1-function ϕ satisfying $\phi(0) = 0$, $\nabla\phi(0) = 0$, $\|\nabla\phi\|_\infty \leq \Lambda_0$, $|\nabla\phi(x) - \nabla\phi(y)| \leq \Lambda_0 |x - y|$,
and an orthonormal coordinate system CS_z with origin at z, such that

$$D \cap B(z, R_0) = \{y \in B(z, R_0) : y_d > \phi(y_1, \dots, y_{d-1}) \text{ in } \mathrm{CS}_z\}.$$

A $C^{1,1}$-open set satisfies interior and exterior ball conditions.

Theorem (L., submitted, 2016)

Assume κ satisfies conditions (1.) - (4.). Let D be a $C^{1,1}$-open set with characteristics (R_0, Λ_0). Then there are constants $C \in (0, \infty)$ and $R_2 \in (R_0/2)$ such that, for any $Q \in \partial D$, any $r \in (0, R_2)$, and any non-negative function u on $\mathbb{R}^d$ which is not identical to zero, $\Delta^{\frac{\alpha}{2}, \kappa}$-harmonic on $D \cap B(Q, r)$ and vanishes continuously on $D^c \cap B(Q, r)$,

$$\frac{u(x)}{u(y)} \leq C \frac{\delta_D(x)^{\alpha/2}}{\delta_D(y)^{\alpha/2}}, \qquad \textit{for all } x, y \in D \cap B(Q, r/4).$$

The constant C depends only on $d, \alpha, R_0, \Lambda_0$, and on the constants C_1, C_2, C_3 from the conditions on κ.

Z.-Q. Chen, X. Zhang (2014):
Heat kernel exists and satisfies short time estimates

$$p_\alpha^\kappa(t,x,y) \le c_1 \frac{t}{(t^{1/\alpha}+|x-y|)^{d+\alpha}}, \quad \forall t \in (0,1], \forall x,y \in \mathbb{R}^d,$$
$$p_\alpha^\kappa(t,x,y) \ge c_2 \frac{t}{(t^{1/\alpha}+|x-y|)^{d+\alpha}}, \quad \forall t \in (0,1], \forall x,y \in \mathbb{R}^d.$$

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Levy system with jump kernel

$$J(x,y) = \mathcal{A}(d,-\alpha)\frac{\kappa(x,y)}{|x-y|^{d+\alpha}}.$$

Dynkin formula: For any bounded open U, $x \in U$, and any $f \in C_c^\infty(\mathbb{R}^d)$,

$$\mathbb{E}_x \int_0^{\tau_U} \Delta^{\frac{\alpha}{2},\kappa} f(X_t)dt = \mathbb{E}_x[f(X_{\tau_U})] - f(x),$$

where τ_U is first exit time from U.

Proposition (Bass, Levin, 2002)

Let $\varepsilon > 0$. *There exist constants* $c = c(d, \alpha, C_2, \varepsilon) \in (0,1)$ *and* $C = C(d, \alpha, C_1) \in (0, \infty)$ *such that for any* $x \in \mathbb{R}^d$, $r \in (0, \infty)$,

$$cr^\alpha \leq \inf_{z \in B(x,(1-\varepsilon)r)} \mathbb{E}_z \tau_{B(x,r)} \leq \sup_z \mathbb{E}_z \tau_{B(x,r)} \leq Cr^\alpha.$$

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Together with the elliptic Harnack inequality, we obtain the following lemma which is well-known in the symmetric case, e.g. Bass, Cranston, 1983.

Lemma

Let $r \in (0, \infty)$ *and* k *a positive integer. Let* $x_1, x_2 \in \mathbb{R}^d$ *such that* $|x_1 - x_2| < 2^k r$*. Then there exists a constant* $C = C(d, \alpha, C_1, C_2) \in (0, \infty)$ *such that any non-negative function* u *on* $\mathbb{R}^d$ *that is harmonic on* $B(x_1, r) \cup B(x_2, r)$ *satisfies*

$$C^{-1} 2^{-k(d+\alpha)} u(x_2) \le u(x_1) \le C 2^{k(d+\alpha)} u(x_2).$$

Lemma

Let D be a bounded $C^{1,1}$-open set with characteristics (R_0, Λ_0). Then there are constants $C_8, C_9 \in (0, \infty)$ and $R_2 \in (0, R_0/2)$ such that, for any $Q \in \partial D$, $r \in (0, R_2)$, and any $x \in D \cap B(Q, r/4)$,

$$\mathbb{P}_x \left(X_{\tau_{D \cap B(Q,r)}} \in \{ |\widetilde{y}| < y_d \text{ in } CS_Q, |y - Q| \in (r, rR_0/R_2) \} \right) \geq C_8 \left(\frac{\delta_D(x)}{r} \right)^{\alpha/2}$$

and

$$\mathbb{P}_x (X_{\tau_{D \cap B(Q,r)}} \in D) \leq C_{13} \left(\frac{\delta_D(x)}{r} \right)^{\alpha/2} .$$

The constants C_8, C_{13} and R_2 depend only on α, d, Λ_0, R_0, C_1, C_2, C_3.

Proof of the upper bound $\mathbb{P}_x(X_{\tau_{D\cap B(Q,r)}} \in D) \leq C_{13}\left(\frac{\delta_D(x)}{r}\right)^{\alpha/2}$.
Step 1:

Lemma

There exists a constant $C_7 = C_7(d, \alpha, C_1, C_2) \in (0, \infty)$ such that, for any $r \in (0, 1]$, and any open sets $U \subset D$ with $D \cap B(Q, r) \subset U$,

$$\mathbb{P}_x(X_{\tau_U} \in D) \leq C_7 r^{-\alpha} \mathbb{E}_x[\tau_U], \quad \forall x \in D \cap B(Q, r/2).$$

Proof.

Case $\kappa \equiv 1$: Song, Wu, 1999.
The general case is proved by K. Kim, P. Kim for symmetric κ. In fact, symmetry is not used directly, conditions (1.) - (4.) on κ are sufficient. The Hölder continuity condition (3.) is crucial. □

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Step 2:

$$\mathbb{E}_x(\tau_{D\cap B(Q,r)}) \le C_9 r^{\alpha/2} \delta_D(x)^{\alpha/2}.$$

Test function method

Aim is to bound expected exit time $\mathbb{E}_x(\tau_{D\cap B(Q,r)})$ by $C_9 r^{\alpha/2}\delta_D(x)^{\alpha/2}$.

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Localize: Set $h^{\alpha/2} := \delta_D^{\alpha/2} 1_{D\cap B(Q,r)}$.

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Convolution by nice functions yields smooth approximations (h_k) of $h^{\alpha/2}$.

Use bare hands to show $|\Delta^{\frac{\alpha}{2},\kappa} h_k(x)| \leq C_{10} r^{-\alpha/2}$.

Put h_k into Dynkin formula

$$\mathbb{E}_x \int_0^{\tau_U} \Delta^{\frac{\alpha}{2},\kappa} h_k(X_t)dt = \mathbb{E}_x[h_k(X_{\tau_U})] - h_k(x),$$

and let $k \to \infty$,

$$\mathbb{E}_x[h^{\alpha/2}(X_{\tau_{D\cap B(Q,r)}})] - C_{10} r^{-\alpha/2}\mathbb{E}_x[\tau_{D\cap B(Q,r)}] \leq h^{\alpha/2}(x) = \delta_D(x)^{\alpha/2}.$$

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$\mathbb{E}_x[h^{\alpha/2}(X_{\tau_{D\cap B(Q,r)}})]$ can be bounded below by $Cr^{-\alpha/2}\mathbb{E}_x[\tau_{D\cap B(Q,r)}]$ using Levy system and some scaling argument.

Lemma

Let D be a $C^{1,1}$-open set with characteristics (R_0, Λ_0). Let f be a bounded function on $\mathbb{R}^d$ such that $|f(x) - f(y)| \leq C_3|x-y|^\rho$ for some $\rho \in (\alpha/2, \alpha)$. Let $Q \in \partial D$, $R \in (0, R_0/2]$, and $h(y) := \delta_D(y) 1_{D \cap B(Q,R)}(y)$. Then there exists a constant $C = C(d, \alpha, \sup_{y \in \mathbb{R}^d} |f(y)|, C_3) \in (0, \infty)$, such that

$$\int_{\mathbb{R}^d} \frac{|h^{\alpha/2}(y) - h^{\alpha/2}(x)||f(y) - f(x)|}{|x-y|^{d+\alpha}} dy \leq C\, R^{-\alpha/2}, \quad \forall x \in D \cap B(Q, R/2).$$

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Applying the lemma with $f = \kappa(x, \cdot)$, we get

$$\begin{aligned} |\Delta^{\frac{\alpha}{2},\kappa} h^{\alpha/2}(x)| &\le |\Delta^{\frac{\alpha}{2},\kappa} h^{\alpha/2}(x) - \Delta^{\frac{\alpha}{2}} h^{\alpha/2}(x)| + |\Delta^{\frac{\alpha}{2}} h^{\alpha/2}(x)| \\ &\le C\, R^{-\alpha/2}. \end{aligned}$$

Similar reasoning yields $|\Delta^{\frac{\alpha}{2},\kappa} h_k(x)| \le C\, R^{-\alpha/2}$.

This completes the proof of $\mathbb{P}_x(X_{\tau_{D\cap B(Q,r)}} \in D) \le C_{13} \left(\frac{\delta_D(x)}{r}\right)^{\alpha/2}$.

Thank you