Boundary Harnack principle for nonsymmetric stable-like operators on $C^{1,1}$-open sets

Janna Lierl

University of Illinois at Urbana-Champaign

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Motivating example

Let \( D := \{ x \in \mathbb{R}^d : x_d > 0 \} \) be the upper half space. The function

\[
h(x) := \begin{cases} 
(x_d)^{\alpha/2}, & \text{if } x_d > 0, \\
0, & \text{if } x_d \leq 0
\end{cases}
\]

is harmonic on \( D \) for the fractional Laplacian \( \Delta^{\frac{\alpha}{2}} \), \( \alpha \in (1, 2) \), and vanishes on \( D^c \). The boundary decay rate is \( h(x) \sim \text{dist}(x, \partial D)^{\alpha/2} \).

Aim: Show that harmonic functions for \( \alpha \)-stable like operators have boundary decay rate \( \delta(x)^{\alpha/2} \), where \( \delta(x) = \text{dist}(x, \partial D) \), on any \( C^{1,1} \)-open set \( D \).

Definition

A Borel measurable function \( u \) on \( \mathbb{R}^d \) is harmonic on an open subset \( D \subset \mathbb{R}^d \) if \( \mathbb{E}_x(|u(X_{\tau_B})|) < \infty \) and

\[
u(x) = \mathbb{E}_x u(X_{\tau_B}), \quad \forall x \in B,
\]

for any open set \( B \) relatively compact in \( D \).
Background

Boundary Harnack principle (BHP):
Let $D$ domain, $Q \in \partial D$. For any $u, v$ positive harmonic functions on $D \cap B(Q, r)$, vanishing on $D^c \cap B(Q, r)$,

$$\frac{u(x)}{v(x)} \leq C \frac{u(y)}{v(y)}, \quad \text{for all } x, y \in D \cap B(Q, cr).$$
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First studied for Laplacian and local operators:

- BHP characterizes uniform domains: Aikawa (2001, 2004),
- uniform domains in Harnack-type Dirichlet spaces: Gyrya, Saloff-Coste (2011)
- weaker formulations of BHP: e.g. Ancona ('07); Bass, Burdzy ('91)

Non-local operators: next slide.
Boundary Harnack principle for non-local operators

**Rotationally invariant stable processes:**
Bogdan (1997): on bounded Lipschitz domains,
Song, Wu (1999): on bounded $k$-fat open sets,
Bogdan, Kulczycki and Kwasnicki (2008): on arbitrary open sets

**Symmetric stable processes:**

rotationally invariant Lévy process (on arbitrary open sets):
Kim, Song, Vondraček (2012).

fractional Laplacian on open sets in fractal spaces: Stos (2006)
Boundary Harnack principle for non-local operators

Rotationally invariant stable processes:
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Bogdan, Kumagai and Kwasnicki (2015): BHP holds on arbitrary open sets in metric measure spaces, for many non-local operators. Includes non-symmetric operators, but requires a dual process.
BHP with explicit decay rates on $C^{1,1}$-open sets:


**censored stable processes**: Bogdan, Burdzy, Chen (2003)

**killed subordinate Brownian motion**: Kim, Song, Vondraček (2012)

**subordinate Brownian motions**: Kim, Song, Vondraček (2014), Kim, Mimica (2014)

**perturbed operators and sums of independent operators**: e.g., by Chen, Kim, Song; Chen et al.
BHP with explicit decay rates on $C^{1,1}$-open sets:

**symmetric stable processes:** Kulczycki (1997), Chen, Song (1998)

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**perturbed operators and sums of independent operators:** e.g., by Chen, Kim, Song; Chen et al.

**censored stable-like processes:** Guan (preprint 2007)
Let $\alpha \in (1, 2)$. 

$\alpha$-stable like operator

$$\Delta^{\alpha/2, \kappa} u(x) := A(d, -\alpha) \lim_{\epsilon \to 0} \int_{\{y \in \mathbb{R}^d : |x-y| > \epsilon\}} \frac{\kappa(x, y)(u(y) - u(x))}{|x - y|^{d+\alpha}} dy,$$

for $x \in \mathbb{R}^d$,

where $A(d, -\alpha) := |\alpha|2^{\alpha-1}\Gamma((d + \alpha)/2)\pi^{-n/2}/\Gamma(1 - \alpha/2)$.

Case $\kappa \equiv 1$: Fractional Laplacian.

In general, $\kappa$ may be non-symmetric.
In general, we assume that $\kappa$ has the following properties:

1. (Uniform boundedness) There exist constants $C_1, C_2 \in (0, \infty)$ such that

$$C_1 \leq \kappa(x, y) \leq C_2, \quad \forall x, y \in \mathbb{R}^d.$$
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2. $\kappa(x, x + y) = \kappa(x, x - y)$ for all $x, y \in \mathbb{R}^d$. 
In general, we assume that $\kappa$ has the following properties:

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2. $\kappa(x, x + y) = \kappa(x, x - y)$ for all $x, y \in \mathbb{R}^d$.

3. There exist constants $C_3 \in (0, \infty)$ and $\rho \in (\alpha/2, \alpha)$ such that

   $$|\kappa(x, y) - \kappa(x, x)| \leq C_3|x - y|^{\rho}, \quad \forall x, y \in \mathbb{R}^d.$$

4. There exist constants $C_4 \in (0, \infty)$ and $\gamma \in (0, 1)$ such that

   $$|\kappa(x, x + z) - \kappa(y, y + z)| \leq C_4|x - y|^\gamma, \quad \forall x, y \in \mathbb{R}^d.$$
Definition

An open set $D \subset \mathbb{R}^d$ is called a $C^{1,1}$-open set with characteristics $(R_0, \Lambda_0)$ if, for each boundary point $z \in \partial D$, there is a $C^1$-function $\phi$ satisfying $\phi(0) = 0$, $\nabla \phi(0) = 0$, $\|\nabla \phi\|_{\infty} \leq \Lambda_0$, $|\nabla \phi(x) - \nabla \phi(y)| \leq \Lambda_0|x - y|$, and an orthonormal coordinate system $CS_z$ with origin at $z$, such that

$$D \cap B(z, R_0) = \{y \in B(z, R_0) : y_d > \phi(y_1, \ldots, y_{d-1}) \text{ in } CS_z\}.$$ 

A $C^{1,1}$-open set satisfies interior and exterior ball conditions.
Theorem (L., submitted, 2016)

Assume $\kappa$ satisfies conditions (1.) - (4.). Let $D$ be a $C^{1,1}$-open set with characteristics $(R_0, \Lambda_0)$. Then there are constants $C \in (0, \infty)$ and $R_2 \in (R_0/2)$ such that, for any $Q \in \partial D$, any $r \in (0, R_2)$, and any non-negative function $u$ on $\mathbb{R}^d$ which is not identical to zero, $\Delta^{\frac{\alpha}{2}, \kappa}$-harmonic on $D \cap B(Q, r)$ and vanishes continuously on $D^c \cap B(Q, r)$,

$$\frac{u(x)}{u(y)} \leq C \frac{\delta_D(x)^{\alpha/2}}{\delta_D(y)^{\alpha/2}}, \quad \text{for all } x, y \in D \cap B(Q, r/4).$$

The constant $C$ depends only on $d, \alpha, R_0, \Lambda_0$, and on the constants $C_1, C_2, C_3$ from the conditions on $\kappa$. 
Heat kernel exists and satisfies short time estimates

\[ p_{\alpha}^\kappa(t, x, y) \leq c_1 \frac{t}{(t^{1/\alpha} + |x - y|)^{d+\alpha}}, \quad \forall t \in (0, 1], \forall x, y \in \mathbb{R}^d, \]

\[ p_{\alpha}^\kappa(t, x, y) \geq c_2 \frac{t}{(t^{1/\alpha} + |x - y|)^{d+\alpha}}, \quad \forall t \in (0, 1], \forall x, y \in \mathbb{R}^d. \]

\[ p_{\alpha}^{\kappa}(t, x, y) \leq c_1 \frac{t}{(t^{1/\alpha} + |x - y|)^{d+\alpha}}, \quad \forall t \in (0, 1], \forall x, y \in \mathbb{R}^d, \]

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Levy system with jump kernel

\[ J(x, y) = \mathcal{A}(d, -\alpha) \frac{\kappa(x, y)}{|x - y|^{d+\alpha}}. \]

Dynkin formula: For any bounded open \( U, x \in U, \) and any \( f \in C_c^\infty(\mathbb{R}^d), \)

\[ \mathbb{E}_x \int_0^{\tau_U} \Delta^{\frac{\alpha}{2}, \kappa} f(X_t) dt = \mathbb{E}_x[f(X_{\tau_U})] - f(x), \]

where \( \tau_U \) is first exit time from \( U. \)
Proposition (Bass, Levin, 2002)

Let $\varepsilon > 0$. There exist constants $c = c(d, \alpha, C_2, \varepsilon) \in (0, 1)$ and $C = C(d, \alpha, C_1) \in (0, \infty)$ such that for any $x \in \mathbb{R}^d$, $r \in (0, \infty)$,

$$cr^\alpha \leq \inf_{z \in B(x, (1 - \varepsilon)r)} \mathbb{E}_z \tau_{B(x, r)} \leq \sup_{z} \mathbb{E}_z \tau_{B(x, r)} \leq Cr^\alpha.$$
Proposition (Bass, Levin, 2002)

Let \( \varepsilon > 0 \). There exist constants \( c = c(d, \alpha, C_2, \varepsilon) \in (0, 1) \) and \( C = C(d, \alpha, C_1) \in (0, \infty) \) such that for any \( x \in \mathbb{R}^d \), \( r \in (0, \infty) \),

\[
    c r^\alpha \leq \inf_{z \in B(x,(1-\varepsilon)r)} \mathbb{E}_z \tau_{B(x,r)} \leq \sup_z \mathbb{E}_z \tau_{B(x,r)} \leq C r^\alpha.
\]

Together with the elliptic Harnack inequality, we obtain the following lemma which is well-known in the symmetric case, e.g. Bass, Cranston, 1983.

Lemma

Let \( r \in (0, \infty) \) and \( k \) a positive integer. Let \( x_1, x_2 \in \mathbb{R}^d \) such that \( |x_1 - x_2| < 2^k r \). Then there exists a constant \( C = C(d, \alpha, C_1, C_2) \in (0, \infty) \) such that any non-negative function \( u \) on \( \mathbb{R}^d \) that is harmonic on \( B(x_1, r) \cup B(x_2, r) \) satisfies

\[
    C^{-1} 2^{-k(d+\alpha)} u(x_2) \leq u(x_1) \leq C 2^{k(d+\alpha)} u(x_2).
\]
Lemma

Let $D$ be a bounded $C^{1,1}$-open set with characteristics $(R_0, \Lambda_0)$. Then there are constants $C_8, C_9 \in (0, \infty)$ and $R_2 \in (0, R_0/2)$ such that, for any $Q \in \partial D$, $r \in (0, R_2)$, and any $x \in D \cap B(Q, r/4)$,

$$
\mathbb{P}_x \left( X_{\tau_D \cap B(Q, r)} \in \{ |\tilde{y}| < y_d \text{ in } CS_Q, |y - Q| \in (r, rR_0/R_2) \} \right) \geq C_8 \left( \frac{\delta_D(x)}{r} \right)^{\alpha/2}
$$

and

$$
\mathbb{P}_x \left( X_{\tau_D \cap B(Q, r)} \in D \right) \leq C_{13} \left( \frac{\delta_D(x)}{r} \right)^{\alpha/2}.
$$

The constants $C_8$, $C_{13}$ and $R_2$ depend only on $\alpha$, $d$, $\Lambda_0$, $R_0$, $C_1$, $C_2$, $C_3$. 
Proof of the upper bound $\mathbb{P}_x (X_{T_{D \cap B(Q, r)}} \in D) \leq C_{13} \left( \frac{\delta_D(x)}{r} \right)^{\alpha/2}$.

Step 1:

Lemma

There exists a constant $C_7 = C_7(d, \alpha, C_1, C_2) \in (0, \infty)$ such that, for any $r \in (0, 1]$, and any open sets $U \subset D$ with $D \cap B(Q, r) \subset U$,

$$\mathbb{P}_x (X_{\tau_U} \in D) \leq C_7 r^{-\alpha} \mathbb{E}_x [\tau_U], \quad \forall x \in D \cap B(Q, r/2).$$

Proof.

Case $\kappa \equiv 1$: Song, Wu, 1999.

The general case is proved by K. Kim, P. Kim for symmetric $\kappa$. In fact, symmetry is not used directly, conditions (1.) - (4.) on $\kappa$ are sufficient. The Hölder continuity condition condition (3.) is crucial.
Proof of the upper bound \( \mathbb{P}_x(X_{\tau_{D \cap B(Q,r)}} \in D) \leq C_{13} \left( \frac{\delta_D(x)}{r} \right)^{\alpha/2} \).

Step 1:

**Lemma**

*There exists a constant \( C_7 = C_7(d, \alpha, C_1, C_2) \in (0, \infty) \) such that, for any \( r \in (0, 1] \), and any open sets \( U \subset D \) with \( D \cap B(Q, r) \subset U \),

\[
\mathbb{P}_x(X_{\tau_U} \in D) \leq C_7 r^{-\alpha} \mathbb{E}_x[\tau_U], \quad \forall x \in D \cap B(Q, r/2).
\]

**Proof.**

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Step 2:

\[
\mathbb{E}_x(\tau_{D \cap B(Q,r)}) \leq C_9 r^{\alpha/2} \delta_D(x)^{\alpha/2}.
\]
Test function method
Aim is to bound expected exit time $E_x(\tau_{D \cap B(Q,r)})$ by $C_9 r^{\alpha/2} \delta_D(x)^{\alpha/2}$. 
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Aim is to bound expected exit time $\mathbb{E}_x(\tau_{D \cap B(Q,r)})$ by $C_9 r^{\alpha/2} \delta_D(x)^{\alpha/2}$.

Recall that $x \mapsto (x_d)^{\alpha/2} = \delta_D^{\alpha/2}(x)$ is $\Delta^{\alpha/2}$-harmonic on upper half space.
Localize: Set $h^{\alpha/2} := \delta_D^{\alpha/2} 1_{D \cap B(Q,r)}$. 
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Convolution by nice functions yields smooth approximations $(h_k)$ of $h^{\alpha/2}$.

Use bare hands to show $|\Delta^{\frac{\alpha}{2},\kappa} h_k(x)| \leq C_{10} r^{-\alpha/2}$.

Put $h_k$ into Dynkin formula

$$E_x \int_0^{\tau_U} \Delta^{\frac{\alpha}{2},\kappa} h_k(X_t) dt = E_x[h_k(X_{\tau_U})] - h_k(x),$$

and let $k \to \infty$,

$$E_x[h^{\alpha/2}(X_{\tau_{D \cap B(Q,r)}})] - C_{10} r^{-\alpha/2} E_x[\tau_{D \cap B(Q,r)}] \leq h^{\alpha/2}(x) = \delta_D(x)^{\alpha/2}.$$
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Aim is to bound expected exit time \( \mathbb{E}_x(\tau_{D \cap B(Q,r)}) \) by \( C_9 r^{\alpha/2} \delta_D(x)^{\alpha/2} \).

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Put \( h_k \) into Dynkin formula
\[
\mathbb{E}_x \int_0^{\tau_U} \Delta^{\alpha/2, \kappa} h_k(X_t) dt = \mathbb{E}_x[h_k(X_{\tau_U})] - h_k(x),
\]
and let \( k \to \infty \),
\[
\mathbb{E}_x[h^{\alpha/2}(X_{\tau_{D \cap B(Q,r)}})] - C_{10} r^{-\alpha/2} \mathbb{E}_x[\tau_{D \cap B(Q,r)}] \leq h^{\alpha/2}(x) = \delta_D(x)^{\alpha/2}.
\]
\( \mathbb{E}_x[h^{\alpha/2}(X_{\tau_{D \cap B(Q,r)}})] \) can be bounded below by \( C r^{-\alpha/2} \mathbb{E}_x[\tau_{D \cap B(Q,r)}] \) using Levy system and some scaling argument.
Lemma

Let $D$ be a $C^{1,1}$-open set with characteristics $(R_0, \Lambda_0)$. Let $f$ be a bounded function on $\mathbb{R}^d$ such that $|f(x) - f(y)| \leq C_3 |x - y|^{\rho}$ for some $\rho \in (\alpha/2, \alpha)$. Let $Q \in \partial D$, $R \in (0, R_0/2]$, and $h(y) := \delta_D(y)1_{D \cap B(Q, R)}(y)$. Then there exists a constant $C = C(d, \alpha, \sup_{y \in \mathbb{R}^d} |f(y)|, C_3) \in (0, \infty)$, such that

$$\int_{\mathbb{R}^d} \frac{|h^{\alpha/2}(y) - h^{\alpha/2}(x)||f(y) - f(x)|}{|x - y|^{d+\alpha}} dy \leq C R^{-\alpha/2}, \quad \forall x \in D \cap B(Q, R/2).$$
Lemma

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$$\int_{\mathbb{R}^d} \frac{|h^{\alpha/2}(y) - h^{\alpha/2}(x)||f(y) - f(x)|}{|x - y|^{d+\alpha}} \, dy \leq C \, R^{-\alpha/2}, \quad \forall x \in D \cap B(Q, R/2).$$

Applying the lemma with $f = \kappa(x, \cdot)$, we get

$$|\Delta^{\frac{\alpha}{2}, \kappa} h^{\alpha/2}(x)| \leq |\Delta^{\frac{\alpha}{2}, \kappa} h^{\alpha/2}(x) - \Delta^{\frac{\alpha}{2}} h^{\alpha/2}(x)| + |\Delta^{\frac{\alpha}{2}} h^{\alpha/2}(x)| \leq C \, R^{-\alpha/2}.$$

Similar reasoning yields $|\Delta^{\frac{\alpha}{2}, \kappa} h_k(x)| \leq C \, R^{-\alpha/2}$.

This completes the proof of $\mathbb{P}_x(X_{\tau_{D \cap B(Q,r)}} \in D) \leq C_{13} \left( \frac{\delta_D(x)}{r} \right)^{\alpha/2}$. 
Thank you