Average conditions for permanence in nonautonomous competitive systems with nonlocal dispersal

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By the nonautonomous competitive system of partial differential equations (PDEs) with dispersal we mean

$$\frac{\partial u_i}{\partial t} = \rho_i(\int_\Omega K_i(x,y)u_i(t,y)dy - u_i(t,x)) + f_i(t,x,u_1\dots,u_N)u_i, t \geq 0,\ x \in \bar\Omega,\ i = 1,\dots,N \qquad \text{(D)}$$

- $u_i(t,x)$ – population density of the i-th species at time t and spatial location $x \in \bar\Omega$,
- $\Omega \subset \mathbb{R}^n$ – bounded habitat,
- $\rho_i > 0$ – the dispersal rate of the ith species,
- $f_i(t,x,u_1,\dots,u_N)$ – local per capita growth rate of the i-th species,
- $K(\cdot,\cdot)$ is nonlocal convolution kernel satisfying the following assumption

(A1) $K_i(\cdot,\cdot) : \bar\Omega \times \bar\Omega \to R$ are C^1- functions, $\int_\Omega K_i(x,y) \leq 1$ for any $x \in \bar\Omega$, $\int_\Omega K_i(x,y) \neq 1$ and there is a $\delta_0 > 0$ such that for any $x \in \bar\Omega$, $K_i(x,y) > 0$ for $y \in \bar\Omega$ and $||x-y|| < \delta_0$

System is *permanent* if there are positive constants $\underline{\delta}_i, \overline{\delta}_i > 0$ such that for each positive solution $u(t,x) = (u_1(t,x),\dots,u_N(t,x))$ there holds

$$\underline{\delta}_i \leq \liminf_{t\to\infty} \frac{u_i(t,x)}{\varphi_i(x)} \leq \limsup_{t\to\infty} \frac{u_i(t,x)}{\varphi_i(x)} \leq \overline{\delta}_i \quad 1 \leq i \leq N$$

We define a *lower average* of a function f_i as

$$m[f_i] := \liminf_{t-s\to\infty} \frac{1}{t-s}\int_s^t \min_{x\in\bar\Omega} f_i(\tau,x,0,\dots,0)d\tau$$

We define a *upper average* of a function f_i as

$$M[f_i] := \limsup_{t-s\to\infty} \frac{1}{t-s}\int_s^t \max_{x\in\bar\Omega} f_i(\tau,x,0,\dots,0)d\tau.$$

(A2) $f_i : [0,\infty) \times \bar\Omega \times [0,\infty)^N \to \mathbb{R}$ $(1 \leq i \leq N)$ as well as their first derivatives $\frac{\partial f_i}{\partial t}$ $(1 \leq i \leq N)$, $\frac{\partial f_i}{\partial u_j}$ $(1 \leq i,j \leq N)$, $\frac{\partial f_i}{\partial x_k}$ $(1 \leq k \leq n,\ 1 \leq i,j, \leq N)$ are continuous. Moreover, the derivatives $\frac{\partial f_i}{\partial u_j}$ $(1 \leq i,j \leq N)$ are bounded and uniformly continuous on sets of the form $[0,\infty) \times \bar\Omega \times B$ where B is a bounded subset of $[0,\infty)^N$.

(A3) The functions $[[0,\infty) \times \bar\Omega] \ni (t,x) \mapsto f_i(t,x,0,\dots,0) \in \mathbb{R}]$, $1 \leq i \leq N$ are bounded. $\frac{\partial f_i}{\partial u_j}(t,x,u) \leq 0$ for all $t \geq 0$ $x \in \bar\Omega$, $u \in [0,\infty)^N$, $1 \leq i,j, \leq N$, $i \neq j$.

(A5) There exist $\underline{b}_{ii} > 0$ such that $\frac{\partial f_i}{\partial u_i}(t,x,u) \leq -\underline{b}_{ii}$ for all $t \geq 0$, $x \in \bar\Omega$ $1 \leq i \leq N$, $u \in [0,\infty)^N$.

(A6) There exist $\overline{b}_{ij} \geq 0$ such that $\frac{\partial f_i}{\partial u_j}(t,x,u) \geq -\overline{b}_{ij}$ for all $t \geq 0$, $x \in \bar\Omega$ $1 \leq i,j \leq N$, $u \in [0,\infty)^N$.

(A7) $m[f_i] > 0$.

Denote by λ_i the principal eigenvalue of the problem

$$\begin{cases} \int_\Omega K_i(x,y)\varphi_i(y)dy - \varphi_i(x) = \lambda_i\varphi_i(x) & \text{on} \quad \Omega \\ u_i = 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

Main Theorem
Assume (A1) through (A7). If

$$m[f_i] > \rho_i\lambda_i + \sum_{\substack{j=1\\ j\neq i}}^N \frac{\overline{b}_{ij}(M[f_j] + \rho_j\lambda_j)}{\underline{b}_{jj}} \quad 1 \leq i \leq N$$

then system (D) is permanent.