LARGE SOLUTIONS FOR A CLASS OF SEMILINEAR INTEGRO-DIFFERENTIAL EQUATIONS WITH CENSORED JUMPS

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Outline

- Main goal of the talk (1 slide).
- Large solutions for local operators.
- Large solutions for the fractional Laplacian.
- Brief summary of our results (no proofs !!).
- A different approach (with new difficulties).

Main goal

We study existence of large solutions,

$$u(x) < +\infty \quad \text{for } x \in \Omega,$$

$$u(x) \to +\infty \quad \text{as } x \to \partial\Omega$$

for equations like

$$-\int_{|z| \le \varrho(x)} [u(x+z) - u(x)]\nu(dz) + u(x)^p \; = 0.$$

Here $x \in \Omega$, a bounded smooth domain in $\mathbb{R}^N$, $p > 1$, and $\varrho : \bar{\Omega} \to \mathbb{R}$ is such that $0 < \varrho(x) \le \mathrm{dist}(x, \partial\Omega)$.

We also obtain uniqueness of the solution in a class of large solutions whose blow-up rate depends on p, α and the rate at which ϱ shrinks near the boundary.

Large solutions

For local second-order operators,

$$-\Delta u(x) + f(u(x)) = 0 \qquad \text{in } \Omega,$$

$$u(x) \to +\infty \quad \text{as } x \to \partial\Omega$$

the story begins with the works of Keller and Osserman (1957). There exists a large solution if and only if the nonlinearity f satisfies the so-called *Keller-Osserman condition*

$$\int_1^{+\infty} \frac{ds}{\sqrt{F(s)}} < +\infty, \qquad \text{where} \quad F(s) = \int_0^s f(t)dt.$$

In particular, when $f(t) = t^p$, the above condition is verified by a superlinear nonlinearity, that is, for $p > 1$.

Large solutions

After Keller and Osserman, a broad variety of very interesting results concerning existence, uniqueness and asymptotic behavior near the boundary for large solutions of second-order reaction-diffusion equations have been obtained in the PDE framework using different techniques, see
Bandle-Marcus,
Brezis,
DelPino,
Loewner,
Marcus-Veron,
Radulescu,
Garcia Melian-Sabina de Lis,
for a nonexhaustive list of references.

Large solutions

In recent years, large solutions for reaction-diffusion problems associated to anomalous diffusion have been subject of study. The typical example of this anomalous diffusive operator is the so-called fractional Laplacian of order $\alpha \in (0, 2)$, defined as

$$(-\Delta)^{\alpha} u(x) = C_{N,\alpha} \mathrm{P.V.} \int_{\mathbb{R}^N} [u(x+z) - u(x)] |z|^{-(N+\alpha)} dz,$$

where $\mathrm{P.V.}$ stands for the principal value and $C_{N,\alpha}$ is a well-known normalizing constant. In this case, the prototype equation takes the form

$$-(-\Delta)^{\alpha} u + u^p = 0 \quad \text{in } \Omega.$$

Large solutions

Typically, in addition to the equation, we must prescribe the value of u in $\Omega^c = \mathbb{R}^N \setminus \Omega$ as an exterior data in order to evaluate u in the fractional Laplacian.

This is the first feature that must be taken into account.

Felmer-Quaas (2011) construct large solutions provided the exterior data blows up at the boundary.

Large solutions

In a subsequent paper Chen-Felmer-Quaas (2015), provided a uniqueness result (given a datum outside Ω) for a class of large solutions with a controlled blow-up rate at the boundary.

Here the blow-up rate of a solution is naturally restricted by the convergence of the integral defining the fractional Laplacian.

Large solutions. Results

The nonlocal operator that will be considered in this talk is

$$I(u,x) = \mathrm{P.V.} \int_{B_{\varrho(x)}} [u(x+z) - u(x)] K(z) |z|^{-(N+\alpha)} dz,$$

for some $\alpha \in (0,2)$.
The functions K, ϱ satisfy the following assumptions:

(M) $K : \mathbb{R}^N \to \mathbb{R}$ is measurable, bounded, radially symmetric function satisfying $K \geq \kappa > 0$ in $\mathbb{R}^N$, for some $\kappa > 0$.

(D) There exists $\sigma \geq 1$ and $0 < \Lambda \leq 1$ such that

$$\varrho(x) = \Lambda [d(x)]^{\sigma}.$$

Large solutions

Theorem. *Assume*

$$\varrho(x) = \Lambda[d(x)]^{\sigma}$$

with p, σ and Λ satisfying one of the following configurations:

(i) Linear Censorship: $\sigma = \Lambda = 1$ *and* $p > 1 + \alpha$*.*

(ii) Linear Strict Censorship: $\sigma = 1, \Lambda < 1$ *and* $p > 1$*.*

(iii) Superlinear Censorship: $1 < \sigma < 2/(2 - \alpha)$ *and* $p > 1$*.*

Then, there exists a classical large solution.

Large solutions

This solution is strictly positive in Ω *and it is the minimal solution in the class of large solutions.*

In addition, there exist constants $0 < \bar{c} < \bar{C}$ *such that*

$$\bar{c}\,[d(x)]^{-\gamma} \le u(x) \le \bar{C}\,[d(x)]^{-\gamma}, \qquad \text{as } x \to \partial\Omega,$$

with

$$\gamma = \frac{\sigma(\alpha-2)+2}{(p-1)}.$$

Large solutions

Moreover, this solution is the unique solution in the class of large solutions v satisfying the boundary blow-up rate

$$0 < \liminf_{\Omega \ni x \to \partial\Omega} v(x)[d(x)]^{\gamma} \leq \limsup_{\Omega \ni x \to \partial\Omega} v(x)[d(x)]^{\gamma} < +\infty.$$

Remark. We don't have nonexistence results. Hence we don't know if these results are sharp.

Large solutions

Note that in cases (ii) and (iii) of the above theorem, the domain of integration in the nonlocal operator “does not touch the boundary”, even when x is sufficiently close to $\partial\Omega$ and for this reason we call them as the “*strictly censored case*”.

The main particularity in this situation is that the problem permits blow-up solutions which are not in $L^1(\Omega)$.

This makes a qualitative difference with the remaining case (i) (which we refer as “*weakly censored case*”). In this case, the integrable blow-up profile at the boundary of the solution is a structural requirement to evaluate the nonlocal operator.

Large solutions

Our second result deals with the precise blow-up rate for a large solution.

Theorem. *Assume* **(M)** *with K continuous at the origin. Let*

$$\varrho(x) = \Lambda[d(x)]^{\sigma}$$

with Λ, σ and p satisfying one of the configurations of the previous Theorem.

Then, the large solution u satisfies the asymptotic behavior

$$\lim_{d(x)\to 0} u(x)[d(x)]^{\gamma} = \bar{C}_0,$$

and $\bar{C}_0 > 0$ is a constant depending on N, Λ, σ, α, p, b_0 and $K(0)$.

Large solutions

Our results are obtained in the framework of viscosity solutions theory for nonlocal problems.

A large solution is obtained as the limit of a sequence of solutions of Dirichlet problems with bounded boundary data tending to infinity. In fact we have that,

there exists a unique viscosity solution $u_R \in C(\bar{\Omega})$ *for the equation satisfying* $u_R(x) = R$ *on* $\partial\Omega$*.*

Now, we just study

$$\lim_{R\to\infty} u_R(x).$$

Large solutions

The reactive term (the nonlinearity) allows us to construct a locally bounded supersolution which implies the approximating sequence is uniformly bounded in $L^\infty_{loc}(\Omega)$.

In fact, there exist $M, L > 0$ such that the function

$$\bar{w}(x) = Md^{-\gamma}(x) + L,$$

is a viscosity supersolution.

The reaction-diffusion nature of the problem and suitable comparison principles provide the required compactness to pass to the limit.

Large solutions

The diffusive term (the elliptic operator) allows to construct an adequate subsolution which “lifts to infinity” the approximating sequence at the boundary.

The diffusive term is also the key ingredient in the application of elliptic regularity results (Caffarelli-Silvestre (2009), Barles.Chasseigne-Imbert (2011)).

Large solutions

We remark that the diffusive part has a *censored or regional nature*. Hence we don't need to consider any datum outside $\overline{\Omega}$.

Note that the character of the problem prevents the use of some second-order methods concerning the analysis of auxiliary problems localized in subdomains, basically because in that case we are qualitatively changing the structure of the problem.

Large solutions

In our case one has to take into account the rate at which the function ϱ shrinks near the boundary.

In fact, the blow-up rate of the constructed large solution is determined both by the singularity of the measure ν $(= \alpha)$, the rate at which $\varrho(x)$ shrinks as $x \to \partial\Omega$ $(= \sigma)$ and, of course, the size of the nonlinearity $(= p)$.

Recall that

$$u(x) \sim [d(x)]^{-\gamma}$$

with

$$\gamma = \frac{\sigma(\alpha - 2) + 2}{(p - 1)}.$$

Large solutions. Open

Show similar results for

$$\text{P.V.} \int_{B_{\varrho(x)}} [u(x+z) - u(x)] K(z) |z|^{-(N+\alpha)} dz,$$

with

$$\varrho(x) = [u(x)]^{-a}.$$

Note that $\rho(x)$ is small when $u(x)$ is large.

Maybe for

$$\varrho(x) = \min\{[u(x)]^{-a},\ k\}.$$

THANKS !!!!.