Intrinsic contractivity properties for relativistic Schrödinger operators

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based on joint work with M. Kwaśnicki and J. Lőrinczi

Classical Schrödinger operators $H = -\Delta + V$

Consider

$$V \in L^\infty_{\text{loc}}(\mathbb{R}^d) \;\text{ such that }\; V(x) \to \infty \text{ as } |x| \to \infty$$

and

$$H = -\Delta + V \quad \text{on} \quad L^2(\mathbb{R}^d).$$

Let $\{e^{-Ht} : t \geq 0\}$ be the corresponding Schrödinger semigroup.

- $\lambda_0 := \inf \operatorname{Spec} H > -\infty$ is a non-degenerate eigenvalue, i.e., a ground state of H exists: $0 < \varphi_0 \in L^2(\mathbb{R}^d) \cap C_b(\mathbb{R}^d)$, $H\varphi_0 = \lambda_0 \varphi_0$
- $e^{-Ht} : L^2(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$, $2 \leq p \leq \infty$, are bounded for $t > 0$

Ground state transformation

Consider the spaces $L^p(\mathbb{R}^d, \mu)$, $p \in [1, \infty]$, where $\mu(dx) = \varphi_0^2(x)dx$, and define the unitary map $M_{\varphi_0} : L^2(\mathbb{R}^d, \mu) \to L^2(\mathbb{R}^d, dx)$ by $M_{\varphi_0} f = f\varphi_0$.

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The operator

$$A := M_{\varphi_0}^{-1}(H - \lambda_0)M_{\varphi_0} \quad \text{on} \quad L^2(\mathbb{R}^d, \mu)$$

generates a semigroup $\left\{e^{-At} : t \geq 0\right\}$ called the ground state transformed (or intrinsic) semigroup. It determines a Markov process with stationary measure μ.

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Example (Harmonic Oscillator and Ornstein-Uhlenbeck operator)

Let $H = -\Delta + |x|^2$. Then $\varphi_0(x) = ce^{-|x|^2/2}$ and for $f \in C_0^2(\mathbb{R}^d)$ one has

$$Af(x) = -\Delta f(x) - 2\nabla \log \varphi_0(x) \cdot \nabla f(x) = -\Delta f(x) + 2x \cdot \nabla f(x), \quad x \in \mathbb{R}^d.$$

Intrinsic contractivity properties

Definition (Contractivity properties). The semigroup $\{e^{-Ht} : t \geq 0\}$ is

(i) intrinsically hypercontractive if $\{e^{-At} : t \geq 0\}$ is hypercontractive:
$\forall p \in (2, \infty)\ \exists t_p > 0\ \forall t \geq t_p \quad e^{-At} : L^2(\mathbb{R}^d, \mu) \to L^p(\mathbb{R}^d, \mu)$ is bounded

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(ii) intrinsically ultracontractive if $\{e^{-At} : t \geq 0\}$ is ultracontractive:
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Example. Consider $H = -\Delta + V$, with $V(x) = |x|^\alpha \log(1 + |x|)^\beta$, $\alpha > 0, \beta \geq 0$.

- $\alpha < 2$: no intrinsic hypercontractivity
- $\alpha = 2,\ 0 \leq \beta \leq 2$: intr. hypercontractivity / no intr. ultracontractivity
- $\alpha > 2$ or $\alpha = 2,\ \beta > 2$: intrinsic ultracontractivity

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[Alziary-Takáč, JFA, 2009] Let $H = -\Delta + V$ with $V(x) = V(|x|)$. Then:

$$\text{intrinsic ultracontractivity} \iff \exists r_0 > 0 \int_{r_0}^{\infty} \frac{1}{\sqrt{V(r)}}\,dr < \infty$$

Large time intrinsic contractivity properties

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(i) intrinsic hypercontractivity:
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Let $H = -\Delta + V$, with $V(x) = |x|^2$. Then:

- intrinsic hypercontractivity holds
- asymptotic intrinsic ultracontractivity does not hold

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Applications. TFAE:

- asymptotic intrinsic ultracontractivity of $\{e^{-Ht} : t \geq 0\}$
- $\exists t_0 > 0 \quad e^{-Ht}(x,y) \asymp e^{-\lambda_0 t}\varphi_0(x)\varphi_0(y), \quad x,y \in \mathbb{R}^d, \quad t \geq t_0$
- $\exists t_0 > 0 \quad \sup_{x,y\in\mathbb{R}^d} \left| \frac{e^{\lambda_0 t} e^{-Ht}(x,y)}{\varphi_0(x)\varphi_0(y)} - 1 \right| \leq Ce^{-(\lambda_1-\lambda_0)t}, \quad t \geq t_0$
- ...

Class of pseudodifferential operators

Let $d \geq 1$ and let L be determined by

$$\widehat{Lf}(\xi) = -\psi(\xi)\widehat{f}(\xi), \quad f \in D(L) := \left\{ f \in L^2(\mathbb{R}^d) : \psi\widehat{f} \in L^2(\mathbb{R}^d) \right\},$$

with

$$\psi(\xi) = \xi \cdot A\xi + \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot z))k(z)dz$$

- $A = (a_{jk})_{1 \leq j,k \leq d}$ – symmetric nonnegative definite $d \times d$ matrix
- $k(z)dz$ is infinite and $\int (1 \wedge |z|^2)k(z)dz < \infty$
- $k(z)$ is symmetric, i.e., $k(x) = k(-x)$
- $\exists$ nonincreasing $g : (0, \infty) \to (0, \infty)$ such that

  $$k(x) \asymp g(|x|), \quad x \in \mathbb{R}^d \setminus \{0\},$$

  and

  $$\sup_{|x| \geq 1} \frac{g_1 \star g_1(|x|)}{g_1(|x|)} < \infty, \quad \text{where} \quad g_1(|x|) = g(|x|)\mathbf{1}_{\{|x| \geq 1\}}$$
- ... $+$ some extra mild regularity assumption on A and g...

Class of pseudodifferential operators - key examples

For $\varphi \in C_c^2(\mathbb{R}^d)$

$$L\varphi(x) = \sum_{j,k=1}^{d} a_{jk} \frac{\partial^2 \varphi}{\partial x_j \partial x_k}(x) + \int_{\mathbb{R}^d} \big(\varphi(x+y) - \varphi(x) - \nabla\varphi(x) \cdot y \mathbf{1}_{\{|y|\le 1\}}\big) k(z) dy$$

Examples

- $L = -(-\Delta)^{\alpha/2},\ \alpha \in (0,2)$ $\Leftarrow k(z) = \frac{C_{d,\alpha}}{|z|^{d+\alpha}}$
- $L = -\big(-\Delta + m^{\frac{2}{\alpha}}\big)^{\frac{\alpha}{2}} + m,\ \alpha \in (0,2),\ m > 0 \Leftarrow k(z) \asymp \frac{e^{-m^{1/\alpha}|z|}}{|z|^{\frac{d+\alpha+1}{2}}},\ |z| \ge 1$
- $L = \Delta - (-\Delta)^{\alpha/2},\ \alpha \in (0,2)$ $\Leftarrow A = \mathrm{Id},\ k(z) = \frac{C_{d,\alpha}}{|z|^{d+\alpha}}$
- $L = -\log(1 + (-\Delta)^{\alpha/2}),\ \alpha \in (0,2)$ $\Leftarrow k(z) \asymp \frac{C_{d,\alpha}}{|z|^{d+\alpha}},\ |z| \ge 1$

Nonlocal Schrödinger operators $H = -L + V$

Let $V(x) \asymp f(|x|)$ for some non-decreasing $f : [0, \infty) \to \mathbb{R}$, $\lim_{r\to\infty} f(r) = \infty$.
$H = -L + V$ has a ground state: $\exists\ 0 < \varphi_0 \in L^2(\mathbb{R}^d) \cap C_b(\mathbb{R}^d)$, $H\varphi_0 = \lambda_0\varphi_0$.

Consider $A := M_{\varphi_0}^{-1}(H - \lambda_0)M_{\varphi_0}$ on $L^2(\mathbb{R}^d, \mu)$, with $\mu(dx) = \varphi_0^2(x)dx$.

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Theorem [K-Kwaśnicki-Lőrinczi, 2015][K-Lőrinczi, Ann. Probab., 2015]

TFAE:

(i) intrinsic hypercontractivity:
$\forall p \in (2, \infty)\ \exists t_p > 0\ \forall t \geq t_p\ e^{-At} : L^2(\mathbb{R}^d, \mu) \to L^p(\mathbb{R}^d, \mu)$ is bounded

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(iii) $\exists C, R > 0\ \forall |x| \geq R\quad V(x) \geq C|\log k(x)|$

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(iv) ground state domination:
$\forall p \in (2, \infty]\ \exists t_p > 0\ \forall t \geq t_p\ \ \frac{e^{-Ht}\mathbf{1}}{\varphi_0} \in L^p(\mathbb{R}^d, \mu)$

L^p-ground state domination and examples

Definition (Ground state domination property) Let $p \in (2, \infty]$.

We say that e^{-Ht} is L^p-ground state dominated (L^p-GSD) if

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Lemma [K-Kwaśnicki-Lőrinczi, 2015] Let $p \in (2, \infty]$. The following hold.

(i) If e^{-Ht} is L^p-GSD, then $e^{-2At} : L^2(\mathbb{R}^d, \mu) \to L^p(\mathbb{R}^d, \mu)$ is bounded.

(ii) If $e^{-At} : L^2(\mathbb{R}^d, \mu) \to L^p(\mathbb{R}^d, \mu)$ is bounded, then e^{-2Ht} is L^p-GSD.

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Example Let $H = (-\Delta + m^{2/\alpha})^{\alpha/2} - m + V$, $\alpha \in (0, 2)$, $m \geq 0$. TFAE:

(i) intrinsic hypercontractivity

(ii) asymptotic intrinsic ultracontractivity

(iii) $\exists C, R > 0 \; \forall |x| \geq R \quad V(x) \geq C \begin{cases} |x| & \text{when} \quad m > 0 \\ \log|x| & \text{when} \quad m = 0 \end{cases}$

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