Gradient perturbations of non-local operators

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Based on joint results with T. Grzywny.

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Introduction - generator

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$$\partial_t \, p_t = L p_t$$

Introduction - formulation of the problem

Fact

For all $s \in \mathbb{R}$, $x \in \mathbb{R}^d$, $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$

$$\int_s^\infty \int_{\mathbb{R}^d} p_{u-s}(dz - x)\Big[\partial_u \phi(u,z) + L\phi(u,z)\Big] du = -\phi(s,x)\,.$$

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Problem

$$\partial_t\, \tilde{p}_t = \widetilde{L}\, \tilde{p}_t\,.$$

Transition density

A function $p\colon \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to [0, \infty)$ is called a transition density if for all $s < u < t$ and $x, y \in \mathbb{R}^d$ if satisfies the Chapman-Kolmogorov equation,

$$\int_{\mathbb{R}^d} p(s, x, u, z) p(u, z, t, y)\, dz = p(s, x, t, y)\,.$$

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Natural assumption on p:
for all $x, y \in \mathbb{R}^d$ and $s < u < t$,

$$\nabla_x p(s, x, t, y) \text{ exists and}$$
$$\nabla_x p(s, x, t, y) = \int_{\mathbb{R}^d} \nabla_x p(s, x, u, z) p(u, z, t, y)\, dz\,,$$

where the integral is absolutely convergent.

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Drift function

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$$p_n(s,x,t,y) = \int_s^t \int_{\mathbb{R}^d} p_{n-1}(s,x,u,z) b(u,z) \cdot \nabla_z p(u,z,t,y) dz du .$$

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Gradient perturbation of p by b

$$\tilde{p}(s,x,t,y) = \sum_{n=0}^{\infty} p_n(s,x,t,y)\,.$$

Condition on the drift

Recall

$$p_1(s,x,t,y) = \int_s^t \int_{\mathbb{R}^d} p(s,x,u,z)b(u,z) \cdot \nabla_z p(u,z,t,y)dzdu\,.$$

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Natural condition

$$\int_s^t \int_{\mathbb{R}^d} p(s,x,u,z) |b(u,z)| |\nabla_z p(u,z,t,y)| \, dz du \le \big[\eta + Q(s,t)\big] p(s,x,t,y)\,,$$

where $\eta \ge 0$ and $Q(s,u) + Q(u,t) \le Q(s,t), \quad s < u < t.$

Modified condition on the drift

Let $C \geq 1$, $T_0 \in (0, \infty]$ be such that for all $x, y \in \mathbb{R}^d$,

$$p(s,x,t,y) \leq C\, p^*(s,x,t,y)\,, \qquad 0 < t-s < T_0\,. \tag{1}$$

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Definition

$b \in \mathcal{N}_\nabla(p, p^*, C, T_0, \eta, Q)$ if p and p^* satisfy (1) and for all $0 < t - s < T_0$ and $x, y \in \mathbb{R}^d$,

$$\int_s^t \int_{\mathbb{R}^d} p^*(s,x,u,z)|b(u,z)||\nabla_z p(u,z,t,y)|\, dzdu \leq \big[\eta + Q(s,t)\big]\, p^*(s,x,t,y)\,.$$

We abbreviate $\mathcal{N}_\nabla(p, p, 1, \infty, \eta, Q)$ do $\mathcal{N}_\nabla(p, \eta, Q)$.

General approach

Theorem

Let $b \in \mathcal{N}_{\nabla}(p, p^*, C, T_0, \eta, Q)$ with $\eta \in [0,1)$. Then $\tilde{p}$ converges absolutely and satisfies Chapman-Kolmogorov equation,

$$\int_{\mathbb{R}^d} \tilde{p}(s,x,u,z)\tilde{p}(u,z,t,y)\,dz = \tilde{p}(s,x,t,y)\,, \quad s < u < t.$$

For all $s < t$, $x, y \in \mathbb{R}^d$, $\varepsilon \in (0, 1-\eta)$,

$$|\tilde{p}(s,x,t,y)| \le \left(\frac{C}{1-\eta-\varepsilon}\right)^{1+Q(s,t)/\varepsilon+(t-s)/T_0} p^*(s,x,t,y)\,.$$

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Let $b \in \mathcal{N}_\nabla(p, p^*, C, T_0, \eta, Q)$ *with* $\eta \in [0,1)$. *Then* $\tilde{p}$ *converges absolutely and satisfies Chapman-Kolmogorov equation,*

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Remark. In particular $p^* = p$, $C = 1$, ($T_0 = \infty$).

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$$\psi(x) = \langle x, Ax\rangle - i\langle x, \zeta\rangle - \int_{\mathbb{R}^d}\left(e^{i\langle x,z\rangle} - 1 - i\langle x, z\rangle \mathbf{1}_{|z|<1}\right)\nu(dz),$$

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$$\widehat{Lf}(\xi) = -\psi(\xi)\hat{f}(\xi)\,.$$

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The density of X_t satisfies

$$p_t(x) = p_t(|x|) \approx \left(t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}}\right),$$
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3G-type inequality, $\alpha \in [1,2)$,

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K. Bogdan, T. Grzywny, M. Ryznar

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Theorem

There is a constant c such that

$$|\nabla_x p_t(x)| \leq c\psi^{-}(1/t)p_t(x), \quad t > 0,\, x \in \mathbb{R}^d.$$

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Assumptions

Generator L corresponding to $(0, \zeta, \nu)$.

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$$\rho_t(x) = \left[\psi_0^-(1/t)\right]^d \wedge \left(\frac{t\psi_0^*(1/|x|)}{|x|^d}\right).$$

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$$\gamma_t(x) \approx \rho_t(x)\,, \qquad t \in (0, T_1),\ x \in \mathbb{R}^d\,.$$

Main results

Let $\mathbb{T} = T_1$ if ν is symmetric, $\mathbb{T} = T_1 \wedge [1/\psi^*(1 \vee \theta_*)]$ otherwise.

$$p(s,x,t,y) \leq Cp^*(s,x,t,y)\,, \qquad 0 < t-s < \mathbb{T},\ x,y \in \mathbb{R}^d\,.$$

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$$\tilde{p}(s,x,t,y) \geq 0\,, \qquad \int_{\mathbb{R}^d} \tilde{p}(s,x,t,z)\,dz = 1\,,$$

$$\int_s^\infty \int_{\mathbb{R}^d} \tilde{p}(s,x,u,z)\left[\partial_u + L + b(u,z)\cdot\nabla_z\right]\phi(u,z)\,dzdu = -\phi(s,x)\,.$$

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for $s + t < 1/[\pi^2\psi_0(\theta_*)]$, $x, y \in \mathbb{R}^d$.

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Kato condition

Assume that if $h \to 0$, then

$$\sup_{\substack{x,y\in\mathbb{R}^d \\ 0<t-s\leq h}} \int_s^t \int_{\mathbb{R}^d} \left(\widehat{p}^*(s,x,u,z) + \widehat{p}^*(u,z,t,y)\right) |b(u,z)| dz du \to 0 \,.$$

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Thank you.