Asymptotics of solutions in nonlocal spatially continuous parabolic Anderson-type models with Poissonian interaction: annealed and quenched behaviour

Katarzyna Pietruska-Pałuba

University of Warsaw

Bedlewo, June 28th, 2016

Joint work with K. Kaleta (arXiv:1601.05597)

Introduction

We will be interested in the long-time behaviour of solutions to parabolic Anderson problem on $\mathbb{R}^d$ (abbr. PAM) driven by the operator L :

$$\begin{aligned} u_t^\omega(t,x) &= L_x u^\omega(t,x) - V^\omega(x)u^\omega(t,x), \quad t>0, \\ u^\omega(0,x) &= 1, \end{aligned}$$

where

- L is the generator of a Lévy process on $\mathbb{R}^d$,
  (L_x indicates that the operator acts on the space variable x),
- V^ω is a random potential of Poissonian type over some probability space $(\Omega, \mathcal{M}, \mathbb{P})$, corresponding to random impurities in the medium.

Introduction

We will be interested in the long-time behaviour of solutions to parabolic Anderson problem on $\mathbb{R}^d$ (abbr. PAM) driven by the operator L :

$$\begin{aligned} u_t^\omega(t,x) &= L_x u^\omega(t,x) - V^\omega(x)u^\omega(t,x), \quad t>0, \\ u^\omega(0,x) &= 1, \end{aligned}$$

where

- L is the generator of a Lévy process on $\mathbb{R}^d$,
  (L_x indicates that the operator acts on the space variable x),
- V^ω is a random potential of Poissonian type over some probability space $(\Omega, \mathcal{M}, \mathbb{P})$, corresponding to random impurities in the medium.

  One is interested in the asymptotics of
  - $\mathbb{E}[u^\omega(t,x)]$ (the annealed behaviour of u^ω),
  - $u^\omega(t,x)$, $\mathbb{P}$-a.s. (the quenched behaviour of u^ω),

  as $t \to \infty$.

Settings – probabilistic

- $X = (X_t)_{t\geq 0}$ – **symmetric Lévy process** in $\mathbb{R}^d$, $d \geq 1$
- $\mathbf{P}_x$, $\mathbf{E}_x$ – measure and expected value of X_t starting at $x \in \mathbb{R}^d$
- characteristic function: $\mathbf{E}^0 e^{i\xi\cdot X_t} = e^{-t\psi(\xi)}$, $t > 0$, where

$$\psi(\xi) = \xi \cdot A\xi + \int_{\mathbb{R}^d\setminus\{0\}} (1 - \cos(\xi \cdot z))\nu(dz)$$

  A – symmetric non-negative definite $d \times d$ matrix (*Gaussian coefficient*)
  $\nu(dz)$ – symmetric m. on $\mathbb{R}^d\setminus\{0\}$, $\int(1 \wedge |z|^2)\nu(dz) < \infty$ (*Lévy measure*)
  (we need some also some mild assumptions concerning ψ).
  ψ is called the Lévy-Khinchine exponent of the process X

- generator: for $\varphi \in C_c^2(\mathbb{R}^d)$,

$$L\varphi(x) = \lim_{t\to 0} \frac{\mathbf{E}_x[\varphi(X_t)] - \varphi(x)}{t},$$

  the limit taken in $L^2(\mathbb{R}^d)$.

Settings – analytic

The generator of the process can be expressed analytically:

$$\widehat{Lf}(\xi) = -\psi(\xi)\widehat{f}(\xi), \quad f \in D(L) := \left\{ f \in L^2(\mathbb{R}^d) : \psi\widehat{f} \in L^2(\mathbb{R}^d) \right\};$$

for $\varphi \in C_c^2(\mathbb{R}^d)$

$$L\varphi(x) = \sum_{j,k=1}^{d} a_{jk} \frac{\partial^2 \varphi}{\partial x_j \partial x_k}(x) + \int_{\mathbb{R}^d} \big(\varphi(x+y) - \varphi(x) - \nabla\varphi(x) \cdot y \mathbf{1}_{\{|y|\le 1\}} \big) \nu(dy).$$

Example: symmetric stable processes

$X^{(\alpha)} = (X_t^{(\alpha)})_{t\geq 0}$ – symmetric α-stable process with L-K exponent $\psi^{(\alpha)}$:

$$\psi^{(\alpha)}(\xi) = \begin{cases} \int_0^\infty \int_{S^{d-1}} \frac{1-\cos(\xi\cdot rz)}{r^{1+\alpha}}\, n(dz)dr & \text{when } \alpha \in (0,2) \\ \xi \cdot A\xi & \text{when } \alpha = 2 \end{cases}$$

n – symmetric finite measure on unit sphere S^{d-1}
$A = (a_{ij})_{1\leq i,j\leq d}$ – symmetric nonnegative definite matrix

When n is the uniform distribution on S^{d-1} for $\alpha \in (0,2)$ or $A \equiv a\mathrm{Id}$ with some $a > 0$ for $\alpha = 2$, then the process is called *isotropic α–stable.*

We assume the nondegeneracy condition $\inf_{|\xi|=1} \psi^{(\alpha)}(\xi) > 0$.

For the isotropic α–stable process, $\alpha \in (0,2)$, the generator is the fractional Laplacian: $L = -(-\Delta)^{\alpha/2}$.

Other examples

- Mixture of stable processes: $\psi(\xi) = a_0|\xi|^2 + \sum_{i=1}^n a_i|\xi|^{\alpha_i}$, $a_0 \geq 0$, $a_i > 0$ for $i = 1, 2, ..., n$, $\alpha_i \in (0, 2)$.
- Geometric stable process with Gaussian component: $\psi(\xi) = \xi \cdot A\xi + \log(1 + |\xi|^\delta)$ with nondegenerate A and $\delta \in (0, 2)$.
- Relativistic $\alpha-$stable processes: $\psi(\xi) = (|\xi|^2 + m^{2/\alpha})^{\alpha/2} - m$, $\alpha \in (0, 2)$, $m > 0$.
- Layered stable processes, tempered stable processes, Lamperti stable processes, truncated stable processes...

Principal eigenvalues

Notation:

$\lambda_1^{(\alpha)}(U)$ – principal eigenvalue of the process $(X_t^{(\alpha)})_{t\geq 0}$ killed on leaving an open and bounded set $U \subset \mathbb{R}^d$

$\big(0 < \lambda_1^{(\alpha)}(U) := \inf \operatorname{spec}(-L_U^{(\alpha)})\big)$

$\lambda_1^{BM}(U)$ – the case of Brownian motion ($A \equiv Id$)

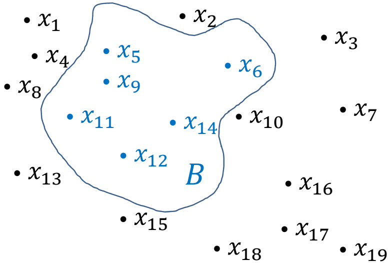
Poisson point process and Poissonian random field

Poisson random measure with intensity $\rho\,\mathrm{d}x$, $\rho > 0$, over $(\Omega, \mathcal{M}, \mathbb{P})$ (the Poisson cloud)

$\mu^\omega(\mathrm{d}y) = \sum_{i=1}^{\infty} \delta_{x_i}(\mathrm{d}y)$
$x_i \sim \omega \in \Omega$
(δ_x – the Dirac point measure at x)

$(\Omega, \mathcal{M}, \mathbb{P})$ – new probability space, different from that of the process X (full stochastic independence)

$\mathbb{E}$ – expected value w.r.t. $\mathbb{P}$



For each Borel set $B \in \mathcal{B}(\mathbb{R}^d)$ with $0 < |B| < \infty$ and $n = 0, 1, 2...$

$\mathbb{P}(\mu^\omega(B) = n) = \exp(-\rho|B|)\dfrac{(\rho|B|)^n}{n!}$ [Poisson distribution with parameter $\rho|B|$],

and for disjoint sets A, B the clouds over A, B are independent.

Poisson point process and Poissonian random field

Poisson random potential

- $W : \mathbb{R}^d \to [0, \infty)$, $W \in L^\infty(\mathbb{R}^d)$, with bounded support (*potential profile*)
- define

$$V^\omega(x) = \int_{\mathbb{R}^d} W(x-y)\mu^\omega(dy) = \sum_{x_i \in \omega} W(x - x_i), \quad x \in \mathbb{R}^d, \quad \omega \in \Omega$$

(*Poissonian potential*)

Poisson point process and Poissonian random field

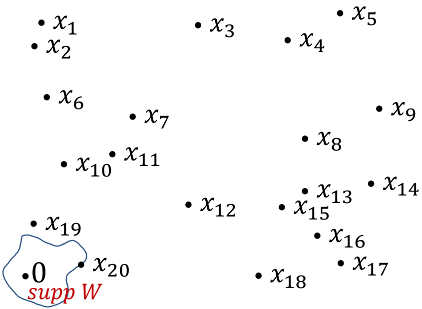
Poisson random potential

- $W : \mathbb{R}^d \to [0, \infty)$, $W \in L^\infty(\mathbb{R}^d)$, with bounded support (*potential profile*)
- define

$$V^\omega(x) = \int_{\mathbb{R}^d} W(x-y)\mu^\omega(dy) = \sum_{x_i \in \omega} W(x - x_i), \quad x \in \mathbb{R}^d, \quad \omega \in \Omega$$

(*Poissonian potential*)

e.g. $W(x) := \mathbf{1}_E(x)$
for some $E \subset \mathcal{B}(\mathbb{R}^d)$



Poisson point process and Poissonian random field

Poisson random potential

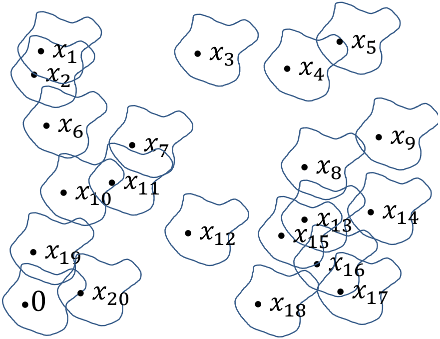
- $W : \mathbb{R}^d \to [0, \infty)$, $W \in L^\infty(\mathbb{R}^d)$, with bounded support (*potential profile*)
- define

$$V^\omega(x) = \int_{\mathbb{R}^d} W(x-y)\mu^\omega(\mathrm{d}y) = \sum_{x_i \in \omega} W(x - x_i), \quad x \in \mathbb{R}^d, \quad \omega \in \Omega$$

(*Poissonian potential*)

e.g. $W(x) := \mathbf{1}_E(x)$
for some $E \subset \mathcal{B}(\mathbb{R}^d)$:

$$V^\omega(x) = \sum_{x_i \in \omega} \mathbf{1}_E(x - x_i)$$
$$= \sum_{x_i \in \omega} \mathbf{1}_{E + x_i}(x)$$



Poisson point process and Poissonian random field

Poisson random potential

- $W : \mathbb{R}^d \to [0, \infty)$, $W \in L^\infty(\mathbb{R}^d)$, with bounded support (*potential profile*)
- define

$$V^\omega(x) = \int_{\mathbb{R}^d} W(x-y)\mu^\omega(dy) = \sum_{x_i \in \omega} W(x - x_i), \quad x \in \mathbb{R}^d, \quad \omega \in \Omega$$

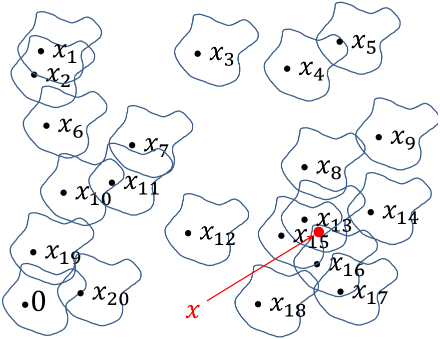
(*Poissonian potential*)

e.g. $W(x) := \mathbf{1}_E(x)$
for some $E \subset \mathcal{B}(\mathbb{R}^d)$

$$V^\omega(x) = \sum_{x_i \in \omega} \mathbf{1}_E(x - x_i)$$

$$= \sum_{x_i \in \omega} \mathbf{1}_{E + x_i}(x)$$

$$V^\omega(x) = \sum_{i \in \{13,15,16\}} \mathbf{1}_{E+x_i}(x) = 3$$



Lévy process in Poissonian environment

For $\omega \in \Omega$, $B \in \mathcal{B}(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and $t > 0$ consider

$$\mathbf{P}_x^{\omega}(X_t \in B) = \mathbf{E}_x[e^{-\int_0^t V^{\omega}(X_s)ds}; X_t \in B] = \mathbf{E}_x[e^{-\sum_{x_i(\omega)} \int_0^t W(X_s - x_i)ds}; X_t \in B]$$

Survival probability of the Lévy process in Poissonian environment

$$u^{\omega}(t,x) := \mathbf{E}_x[e^{-\int_0^t V^{\omega}(X_s)ds}], \quad t > 0, \quad x \in \mathbb{R}^d$$

Lévy process in Poissonian environment

For $\omega \in \Omega$, $B \in \mathcal{B}(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and $t > 0$ consider

$$\mathbf{P}_x^{\omega}(X_t \in B) = \mathbf{E}_x[e^{-\int_0^t V^{\omega}(X_s)ds}; X_t \in B] = \mathbf{E}_x[e^{-\sum_{x_i(\omega)} \int_0^t W(X_s - x_i)ds}; X_t \in B]$$

Survival probability of the Lévy process in Poissonian environment

$$u^{\omega}(t,x) := \mathbf{E}_x[e^{-\int_0^t V^{\omega}(X_s)ds}], \quad t > 0, \quad x \in \mathbb{R}^d$$

.

The function u^{ω} is the solution of (Feynman-Kac formula)

$$\frac{\partial u}{\partial t} = -H^{\omega} u, \quad u(0,x) \equiv 1,$$

where

$$H^{\omega} = -L + V^{\omega}$$

Brownian motion in Poissonian medium: $H^\omega = -\Delta + V^\omega$

Annealed behaviour [Donsker-Varadhan, CPAM, 1975]

$$u^\omega(t,x) = \mathbb{E}\,\mathbf{E}_x[e^{-\int_0^t V^\omega(X_s)ds}] = \exp\left(-Ct^{d/(d+2)}(1+o(1))\right), \quad \text{as} \quad t \to \infty,$$

with

$$C = (\rho\omega_d)^{\frac{2}{d+2}} \left(\frac{d+2}{2}\right) \left(\frac{2\lambda_1^{BM}(B(0,1))}{d}\right)^{\frac{d}{d+2}}$$

ω_d – the volume of a unit ball

Quenched behaviour [Sznitman, PTRF, 1993]

$$u^\omega(t,x) = \exp\left(-C\frac{t}{(\log t)^{2/d}}(1+o(1))\right), \quad \text{as} \quad t \to \infty, \quad \mathbb{P}-\text{a.s.},$$

with

$$C = \left(\frac{\rho\omega_d}{d}\right)^{\frac{2}{d}} \lambda_1^{BM}(B(0,1))$$

Quenched behaviour obtained from annealed one [R. Fukushima, ECP, 2009]

Lévy process with jumps: $H^\omega = -L + V^\omega$

Annealed behaviour [Okura, Z. Wahrscheinlichkeitstheorie verw. Gebiete, 1981]

If

$$\textbf{(A)} \qquad \psi(\xi) = \psi^{(\alpha)}(\xi) + o(|\xi|^\alpha), \quad |\xi| \to 0,$$

for some stable process $X^{(\alpha)}$ with $\alpha \in (0,2]$, and

$$\frac{\psi(\xi)}{(\log|\xi|)^2} \to \infty \quad \text{as} \quad |\xi| \to \infty,$$

then

$$u^\omega(t,x) = \exp\left(-Ct^{d/(d+\alpha)}(1+o(1))\right), \quad \text{as} \quad t \to \infty,$$

with

$$C = (\rho\omega_d)^{\frac{\alpha}{d+\alpha}} \left(\frac{d+\alpha}{\alpha}\right) \left(\frac{2\lambda_{(\alpha)}}{d}\right)^{\frac{d}{d+\alpha}},$$

where

$$\lambda_{(\alpha)} = \inf_{U \text{ open}, |U|=\omega_d} \lambda_1^{(\alpha)}(U).$$

Lévy process with jumps: $H^\omega = -L + V^\omega$

Annealed behaviour [Okura, Z. Wahrscheinlichkeitstheorie verw. Gebiete, 1981]

If

$$\textbf{(A)} \qquad \psi(\xi) = \psi^{(\alpha)}(\xi) + o(|\xi|^\alpha), \quad |\xi| \to 0,$$

for some stable process $X^{(\alpha)}$ with $\alpha \in (0,2]$, and

$$\frac{\psi(\xi)}{(\log|\xi|)^2} \to \infty \quad \text{as} \quad |\xi| \to \infty,$$

then

$$u^\omega(t,x) = \exp\left(-Ct^{d/(d+\alpha)}(1+o(1))\right), \quad \text{as} \quad t \to \infty,$$

with

$$C = (\rho\omega_d)^{\frac{\alpha}{d+\alpha}} \left(\frac{d+\alpha}{\alpha}\right) \left(\frac{2\lambda_{(\alpha)}}{d}\right)^{\frac{d}{d+\alpha}},$$

where

$$\lambda_{(\alpha)} = \inf_{U \text{ open}, |U|=\omega_d} \lambda_1^{(\alpha)}(U).$$

What about the quenched behaviour for jump processes?

The quenched behaviour. The upper bound.

(U) There exist constants $C > 0$, $r_0 \geq 1$, $t_1 \geq t_0 \vee 1$ and a nonincreasing function $F : [0,\infty) \to [0,\infty)$ such that $\lim_{r\to\infty} F(r) = 0$, for which

$$P_0(|X_t| \geq r) \leq C\, t \left(F(r) \vee e^{-r}\right), \quad r \geq r_0 \vee 2t, \quad t \geq t_1.$$

Theorem (KK and KPP, 2016)

Suppose that X is a Lévy process satisfying **(A)** *and* **(U)**. *Then there exist a function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that for every $x \in \mathbb{R}^d$*

$$\limsup_{t\to\infty} \frac{\log u^{\omega}(t,x) - \frac{d}{2}\log h(t)}{g(t)} \leq -\left(\frac{\rho}{d}\right)^{\alpha/d} \lambda_1^{(\alpha)}, \quad \mathbb{Q} - a.s.,$$

where $g(t) = t/(\log h(t))^{\alpha/d}$.

Recall that $\lambda_1^{(\alpha)} = \inf_{U \text{open}, |U|=\omega_d} \lambda_1^{(\alpha)}(U)$.

The rate function

What are the correction term h and the rate function g?

For fixed $\kappa > 0$, $\alpha \in (0,2]$ let $f_{\alpha,\kappa} : [1,\infty) \to [0,\infty)$ be given by

$$f_{\alpha,\kappa}(r) = \left((r \wedge |\log(1 \wedge F(r))|) + \frac{d}{2}\log r \right) \left(\frac{d \log r}{\kappa} \right)^{\frac{\alpha}{d}}.$$

Let $h_{\alpha,\kappa} = f_{\alpha,\kappa}^{-1}$.

The rate function

What are the correction term h and the rate function g?

For fixed $\kappa > 0$, $\alpha \in (0,2]$ let $f_{\alpha,\kappa} : [1,\infty) \to [0,\infty)$ be given by

$$f_{\alpha,\kappa}(r) = \left(\left(r \wedge |\log(1 \wedge F(r))|\right) + \frac{d}{2}\log r \right) \left(\frac{d \log r}{\kappa} \right)^{\frac{\alpha}{d}}.$$

Let $h_{\alpha,\kappa} = f_{\alpha,\kappa}^{-1}$.

It is well defined since $f_{\alpha,\kappa}(1) = 0, \quad \lim_{r\to\infty} f_{\alpha,\kappa}(r) = \infty$, moreover $f_{\alpha,\kappa}(r)$ is strictly increasing in r.

The rate function

What are the correction term h and the rate function g?

For fixed $\kappa > 0$, $\alpha \in (0,2]$ let $f_{\alpha,\kappa} : [1,\infty) \to [0,\infty)$ be given by

$$f_{\alpha,\kappa}(r) = \left((r \wedge |\log(1 \wedge F(r))|) + \frac{d}{2}\log r \right) \left(\frac{d \log r}{\kappa} \right)^{\frac{\alpha}{d}} .$$

Let $h_{\alpha,\kappa} = f_{\alpha,\kappa}^{-1}$.

It is well defined since $f_{\alpha,\kappa}(1) = 0$, $\quad \lim_{r\to\infty} f_{\alpha,\kappa}(r) = \infty$, moreover $f_{\alpha,\kappa}(r)$ is strictly increasing in r. Moreover,

$$t \left(\frac{\kappa}{d \log h_{\alpha,\kappa}(t)} \right)^{\frac{\alpha}{d}} = \Big(h_{\alpha,\kappa}(t) \wedge \left|\log\left(1 \wedge F(h_{\alpha,\kappa}(t))\right)\right| \Big) + \frac{d}{2} \log h_{\alpha,\kappa}(t).$$

The rate function

What are the correction term h and the rate function g?

For fixed $\kappa > 0$, $\alpha \in (0,2]$ let $f_{\alpha,\kappa} : [1,\infty) \to [0,\infty)$ be given by

$$f_{\alpha,\kappa}(r) = \left(\left(r \wedge |\log(1 \wedge F(r))|\right) + \frac{d}{2} \log r \right) \left(\frac{d \log r}{\kappa} \right)^{\frac{\alpha}{d}} .$$

Let $h_{\alpha,\kappa} = f_{\alpha,\kappa}^{-1}$.

It is well defined since $f_{\alpha,\kappa}(1) = 0$, $\lim_{r\to\infty} f_{\alpha,\kappa}(r) = \infty$, moreover $f_{\alpha,\kappa}(r)$ is strictly increasing in r. Moreover,

$$t \left(\frac{\kappa}{d \log h_{\alpha,\kappa}(t)} \right)^{\frac{\alpha}{d}} = \left(h_{\alpha,\kappa}(t) \wedge \left|\log\left(1 \wedge F(h_{\alpha,\kappa}(t))\right)\right| \right) + \frac{d}{2} \log h_{\alpha,\kappa}(t).$$

We take $h = h_{\alpha,\kappa_0}$ with $\kappa_0 = \rho(\lambda_1^{(\alpha)})^{d/\alpha}$.

The upper bound – special cases

(U) There exist constants $C > 0$, $r_0 \geq 1$, $t_1 \geq t_0 \vee 1$ and a nonincreasing function $F : [0, \infty) \to [0, \infty)$ such that $\lim_{r\to\infty} F(r) = 0$, for which

$$P_0(|X_t| \geq r) \leq C\, t\, \left(F(r) \vee e^{-r}\right), \quad r \geq r_0 \vee 2t, \quad t \geq t_1.$$

Corollary

Suppose that the function F appearing in **(U)** *satisfies* $\lim_{t\to\infty} \frac{|\log F(r)|}{\log r} = +\infty$. *Then* $\lim_{t\to\infty} \frac{h(t)}{g(t)} = 0$, *so that for every* $x \in \mathbb{R}^d$

$$\limsup_{t\to\infty} \frac{\log u^{\omega}(t,x)}{g(t)} \leq -\left(\frac{\rho}{d}\right)^{\alpha/d} \lambda_1^{(\alpha)}, \quad \mathbb{P} - a.s..$$

For stable processes with $\alpha \in (0,2)$ and for mixtures of stable processes with $\alpha_i \in (0,2)$, one has $g(t) = ct^{\frac{d}{d+\alpha}}$. For the relativistic process and for the Brownian motion, $g(t) = ct/(\log t)^{2/d}$.

Quenched behaviour. The lower bound

Let $B_R =: [-R, R]^d$ and let $p^{B_R}(t, x, y)$ be the Dirichlet kernel of the process X in B_R. Let

$$G(R) := \inf_{|y| \leq R/2} p^{B_R}(1, 0, y).$$

Theorem (KK and KPP, 2016)

Let X be a Lévy process with exponent ψ satisfying **(A)** *and let the potential profile W be of finite range. Then for any $x \in \mathbb{R}^d$, $\mathbb{Q}-$almost surely*

$$\liminf_{t\to\infty} \frac{\log u^{\omega}(t,x) + 3d \log h(t) + \left|\log G\left(\frac{2h(t)}{(\log h(t))^{\frac{\alpha}{d}+2}}\right)\right|}{g(t)} \geq -\left(\frac{\omega_d \rho}{d}\right)^{\alpha/d} \lambda_1^{(\alpha)}(B(0,1)),$$

where $h(t)$, $g(t)$ are as before, and $\lambda_1^{(\alpha)}(B(0,1))$ is the principal Dirichlet eigenvalue of $B(0,1)$ for the $\alpha-$stable process from **(A)**.

Quenched behaviour for isotropic stable process

Theorem (KK and KPP, 2016)

If

$$\psi(\xi) = c|\xi|^{\alpha}, \quad \xi \in \mathbb{R}^d,$$

for some $\alpha \in (0,2)$ *then for any* $x \in \mathbb{R}^d$,, $\mathbb{P}$*-almost surely,*

$$\exp\left(-C_L t^{d/(d+\alpha)}(1+o(1))\right) \leq u^{\omega}(t,x) \leq \exp\left(-C_U t^{d/(d+\alpha)}(1+o(1))\right)$$

as $t \to \infty$, *with*

$$C_L = \frac{4\alpha + 9d}{2} \left(\frac{2}{d+2\alpha}\right)^{\frac{d}{d+\alpha}} \left(\frac{\rho\omega_d}{d}\right)^{\frac{\alpha}{d+\alpha}} \left(\lambda_1^{(\alpha)}(B(0,1))\right)^{\frac{d}{d+\alpha}}$$

and

$$C_U = \alpha \left(\frac{2}{d+2\alpha}\right)^{\frac{d}{d+\alpha}} \left(\frac{\rho\omega_d}{d}\right)^{\frac{\alpha}{d+\alpha}} \left(\lambda_1^{(\alpha)}(B(0,1))\right)^{\frac{d}{d+\alpha}} .$$

Quenched behaviour for relativistic stable processes

Theorem (KK and KPP, 2016)

Let X_t be the $\alpha-$stable relativistic process, i.e. the process with Lévy-Khinchine exponent $\psi(\xi) = (|\xi|^2 + m^{2/\alpha})^{\alpha/2} - m$, $m > 0$, $\alpha \in (0, 2)$. Then for any $x \in \mathbb{R}^d$, $\mathbb{P}-$almost surely

$$u^\omega(t, x) = \exp(-C_{m,\alpha}(t/(\log t)^{2/d})(1 + o(1))),$$

where the constant $C_{m,\alpha}$ is given by

$$C_{m,\alpha} = \frac{\alpha}{2} m^{1-\frac{2}{\alpha}} \alpha_1^{(2)}(B(0,1)) \left(\frac{\rho\omega_d}{d}\right)^{\frac{d}{2}}.$$

- Observe that when $\alpha \to 2$, then the constant tends to $\alpha_1^{BM}(B(0,1)) \left(\frac{\rho\omega_d}{d}\right)^{d/2}$, which is the constant for the Brownian motion.

Annealed and quenched asymptotics

Quenched asymptotics depends on the rate of decay of the Lévy measure at ∞.

$\nu(\|x\|)$, $\|x\| \to \infty$	**parameters**	**quenched rate**	**annealed rate**
$\frac{C}{\|x\|^{d+\alpha}}$	$\alpha \in (0,2)$	$t^{\frac{d}{d+\alpha}}$	$t^{\frac{d}{d+\alpha}}$
$\frac{C}{\|x\|^{d+\delta}}$	$\delta > 2$	$t^{\frac{d}{d+2}}$	$t^{\frac{d}{d+2}}$
$\frac{C}{e^{\theta(\log \|x\|)^{\beta}}}$	$\theta>0$ $\beta>1$	$t^{\frac{d\beta}{d\beta+2}}$	$t^{\frac{d}{d+2}}$
$\frac{C}{e^{\theta \|x\|^{\beta}}}$	$\theta>0$ $\beta>0$	$\frac{t}{(\log t)^{2/d}}$	$t^{\frac{d}{d+2}}$
0		$\frac{t}{(\log t)^{2/d}}$	$t^{\frac{d}{d+2}}$
BM		$\frac{t}{(\log t)^{2/d}}$	$t^{\frac{d}{d+2}}$

Bibliography

- M.D. Donsker, S.R.S. Varadhan, *Asymptotics for the Wiener sausage*, Comm. Pure Appl. Math. 28 (1975), no. 4, 525–565.
- R. Fukushima, *F*rom the Lifshitz tail to the quenched survival asymptotics in the trapping problem, Electron. Commun. Probab. 14 (2009), 435–446.
- H. Okura, *On the spectral distributions of certain integro-differential operators with random potential*, Osaka J. Math. 16 (1979), no. 3, 633-666.
- H. Okura, *Some limit theorems of Donsker-Varadhan type for Markov processes expectations*, Z. Wahrscheinlichkeitstheorie verw. Gebiete 57 (1981), 419-440.
- A.S. Sznitman, *Brownian asymptotics in a Poissonian environment.* Probab. Theory Relat. Fields 95 (1993), 155-174.
- K. Kaleta, K. Pietruska-Pałuba, *The quenched asymptotics for nonlocal Schrödinger operators with Poissonian potentials*, preprint 2016, available at arXiv:1601.05597.
- K. Kaleta, K. Pietruska-Pałuba, *Integrated density of states for Poisson-Schrödinger processes on the Sierpiński gasket*, Stochastic Process. Appl. 125 (4), 2015, 1244-1281.
- K. Kaleta, K. Pietruska-Pałuba, *Lifschitz singularity for subordinate Brownian motions in presence of the Poissonian potential on the Sierpiński gasket*, preprint 2014, available at arXiv:1410.1715.