Long-time behaviour for nonlocal convection-diffusion problems

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Few words about local diffusion problems

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^d \times (0,\infty), \\ u(0) = u_0. \end{cases}$$

For any $u_0 \in L^1(\mathbb{R})$ the solution $u \in C([0,\infty), L^1(\mathbb{R}^d))$ is given by:

$$u(t,x) = (G(t,\cdot) * u_0)(x)$$

where

$$G(t,x) = (4\pi t)^{-d/2} \exp(-\frac{|x|^2}{4t})$$

Smoothing effect

$$u \in C^\infty((0,\infty), \mathbb{R}^d)$$

Decay of solutions, $1 \leq p \leq q \leq \infty$:

$$\|u(t)\|_{L^q(\mathbb{R}^d)} \lesssim t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \|u_0\|_{L^p(\mathbb{R}^d)}$$

Asymptotics

Theorem

For any $u_0 \in L^1(\mathbb{R}^d)$ *and* $p \geq 1$ *we have*

$$t^{\frac{d}{2}(1-\frac{1}{p})}\|u(t) - MG_t\|_{L^p} \to 0,$$

where $M = \int u_0$.

Proof:

$$(G_t * u_0)(x) - G_t(x)\int u_0 = \frac{1}{(4\pi t)^{d/2}}\int_{\mathbb{R}^d}(\exp(-\frac{|x-y|^2}{4t}) - \exp(-\frac{|x|^2}{4t}))u_0(y)$$

A linear nonlocal problem

E. Chasseigne, M. Chaves and J. D. Rossi, *Asymptotic behavior for nonlocal diffusion equations*, J. Math. Pures Appl., 86, 271–291, (2006).

$$\begin{cases} u_t(x,t) &= J * u - u(x,t) = \int_{\mathbb{R}^d} J(x-y)u(y,t)\,dy - u(x,t), \\ &= \int_{\mathbb{R}^d} J(x-y)(u(y,t)-u(x,t))dy \\ u(x,0) &= u_0(x), \end{cases}$$

where $J : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative, radial function with $\int_{\mathbb{R}^N} J(r)dr = 1$

Models

Gunzburger's papers

1. Analysis and approximation of nonlocal diffusion problems with volume constraints
2. A nonlocal vector calculus with application to nonlocal boundary value problems

Case 1: $s \in (0,1)$,

$$\frac{c_1}{|y-x|^{d+2s}} \leq J(x,y) \leq \frac{c_2}{|y-x|^{d+2s}}$$

Case 2: essentially J is a nice function

Heat equation and nonlocal diffusion

- Similarities
  - bounded stationary solutions are constant
  - a maximum principle holds for both of them

- Difference
  - there is no regularizing effect in general

The fundamental solution can be decomposed as

$$e^{-t}\delta_0(x) + v(x,t), \tag{1}$$

with $v(x,t)$ smooth

$$\begin{aligned} S(t)\varphi = e^{-t}\varphi + v * \varphi &= \text{smooth as initial data} + \text{smooth part} \\ &= \text{no smoothing effect} \end{aligned}$$

Asymptotic Behaviour

• If $\hat{J}(\xi) = 1 - A|\xi|^2 + o(|\xi|^2), \xi \sim 0$, the asymptotic behavior is the same as the one for solutions of the heat equation

$$\lim_{t\to+\infty} t^{d/2} \max_x |u(x,t) - v(x,t)| = 0,$$

where v is the solution of $v_t(x,t) = A\Delta v(x,t)$ with initial condition $v(x,0) = u_0(x)$.

• In $d = 1$ when $J(x) = \frac{1}{2}e^{-|x|}$, or even $J \in L^1(\mathbb{R}, 1+|x|^2)$ we have

$$\lim_{r\to\infty} t^{1/2} \max_{t\in\mathbb{R}} \Big| u(x,t) - (\int_{\mathbb{R}} u_0) G_t(x) \Big| = 0$$

Other results on the linear problem

- I.L. Ignat and J.D. Rossi, *Refined asymptotic expansions for nonlocal diffusion equations*, Journal of Evolution Equations 2008.
- I.L Ignat, J.D. Rossi, JMPA2009, $J(x,y)$
- I.L. Ignat, J.D. Rossi, A. San Antolin, JDE 2012. $J(x,y)=\psi(x-a(y))+\psi(y-a(x))$
- I.L. Ignat, D. Pinasco, J.D. Rossi, A. San Antolin, JAM 2013.

A nonlinear model: convection-diffusion

For $q \geq 1$

$$\begin{cases} u_t - \Delta u + (|u|^{q-1}u)_x = 0 \text{ in } (0,\infty)\times\mathbb{R} \\ u(0) = u_0 \end{cases}$$

- Asymptotic Behaviour by using

$$\frac{d}{dt}\int_{\mathbb{R}} |u|^p dx = -\frac{4(p-1)}{p}\int_{\mathbb{R}} |\nabla(|u|^{p/2})|^2 dx.$$

M. Schonbek, *Uniform decay rates for parabolic conservation laws*, Nonlinear Anal., 10(9), 943–956, (1986).

M. Escobedo and E. Zuazua, *Large time behavior for convection-diffusion equations in* $\mathbb{R}^N$, J. Funct. Anal., 100(1), 119–161, (1991).

The first term in the asymptotic behaviour

With $M=\int u_0$, Escobedo & Zuazua JFA '91 proved

$q>2$,

$$\lim_{t\to\infty} t^{1/2(1-1/p)}\|u(t)-MG_t\|_{L^p(\mathbb{R})}=0$$

$q=2$,

$$\lim_{t\to\infty} t^{1/2(1-1/p)}\|u(t)-f_M(x,t)\|_{L^p(\mathbb{R})}=0$$

where $f_M(x,t)=t^{-1/2}f_M(\frac{x}{\sqrt{t}},1)$ is the unique solution of the viscous Burgers equation

$$\begin{cases} U_t=U_{xx}-(U^2)_x \\ U(0)=M\delta_0 \end{cases}$$

$1<q<2$, Escobedo, Vazquez, Zuazua, ARMA '93

$$\lim_{t\to\infty} t^{1/q(1-1/p)}\|u(t)-U_M(x,t)\|_{L^p(\mathbb{R})}=0$$

Some ideas of the proof

• For $q > 2$

$$u(t) = S(t)u_0 + \int_0^t S(t-s)(u^q)_x(s)ds$$

and use that the nonlinear part decays faster than the linear one

• $q = 2$ scaling: introduce $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$, write the equation for u_λ and observe that the estimates for u are equivalent to the fact that

$$u_\lambda(x,1) \to f_M(x) \text{ in } L^1(\mathbb{R})$$

where f_M is a solution of the viscous Bourgers equation with initial data $M\delta_0$

Main idea: in some moment you have to use compactness

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Proof: the so-called "four step method" :

- scaling - write the equation for u_λ
- estimates and compactness of $\{u_\lambda\}$
- passage to the limit
- identification of the limit

- $1 < q < 2$, read EVZ's paper, entropy solutions, etc...

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Nonlocal Convection-Diffusion

L.I. Ignat and J.D. Rossi, *A nonlocal convection-diffusion equation*, J. Funct. Anal., 251, 399–437, (2007).

$$\begin{cases} u_t(t,x) = (J_1 * u - u)(t,x) + (J_2 * (f(u)) - f(u))(t,x), & t>0,\ x\in\mathbb{R}, \\ u(0,x) = u_0(x), & x\in\mathbb{R}. \end{cases}$$

- J_1 and J_2 are nonnegatives and verify $\int_{\mathbb{R}^d} J_1(x)dx = \int_{\mathbb{R}^d} J_2(x)dx = 1$.
- J_1 even function
- $f(u) = |u|^{q-1}u$ with $q > 1$
- $q > 2$ similar estimates as in the local case
- what about the case $q = 2$?

Similar work

P. Laurençot, Asymptotic Analysis '05, considered the following model for radiating gases

$$\begin{cases} u_t + (\frac{u^2}{2})_x = K * u - u \quad \text{in } (0,\infty)\times\mathbb{R} \\ u(0) = u_0 \end{cases}$$

where $K(x) = e^{-|x|}/2$.

• take care in defining the solutions, entropy solutions, Schochet and Tadmor, ARMA 1992, Lattanzio & Marcati JDE 2003, D. Serre, Scalar conservation laws, etc...

• good news: asymptotic behaviour by scaling

• bad news: Oleinik estimate for u: $u_x \leq \frac{1}{t}$

In a joint work with A. Pazoto we have considered the model (DCDS-A 2014)

$$\begin{cases} u_t = J * u - u + (|u|^{q-1}u)_x, & x \in \mathbb{R}, t > 0 \\ u(0) = \varphi. \end{cases} \tag{2}$$

$$\frac{d}{dt}\int_{\mathbb{R}} u^2(t,x)dx + \iint_{\mathbb{R}^2} J(x-y)(u(x)-u(y))^2 dxdy = 0$$

Question: for $q > 2$ may we obtain similar results as in the case of the classical convection-diffusion: leading term given by the heat kernel?

In a joint work with T. Ignat & D. Stancu (SIAM JMA 2015) we have considered the model

$$\begin{cases} u_t = J * u - u + G * |u|u - |u|u, \quad x \in \mathbb{R}, t > 0 \\ u(0) = \varphi. \end{cases} \tag{3}$$

Big problem: The convection is nonlocal.
Question: may we obtain the asymptotic behaviour without having an Oleinik's estimate?

$$\frac{d}{dt}\int_{\mathbb{R}} u^2(t,x)dx + \iint_{\mathbb{R}^2} J(x-y)(u(x)-u(y))^2 dxdy \leq 0$$

The answer: YES by scaling :)

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The answer: YES by scaling :)

We introduce the scaled functions

$$u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x) \quad \text{and} \quad J_\lambda(x) = \lambda J(\lambda x).$$

Then u_λ satisfies the system

$$\begin{cases} u_{\lambda,t} = \lambda^2 (J_\lambda * u_\lambda - u_\lambda) + \lambda (G_\lambda * u_\lambda^2 - u_\lambda^2), & x \in \mathbb{R}, t > 0 \\ u_\lambda(0,x) = \varphi_\lambda(x) = \lambda \varphi(\lambda x), & x \in \mathbb{R}. \end{cases} \tag{4}$$

Question: four step method?

Estimates on u_λ

There exists $M = M(t_1, t_2, \|\varphi\|_{L^1(\mathbb{R})}, \|\varphi\|_{L^\infty(\mathbb{R})})$ such that

$$\|u_\lambda\|_{L^\infty(t_1,t_2,L^2(\mathbb{R}))} \leq M, \tag{5}$$

$$\lambda^2 \int_{t_1}^{t_2} \int_{\mathbb{R}} \int_{\mathbb{R}} J_\lambda(x-y)(u_\lambda(x) - u_\lambda(y))^2 dx dy \leq M. \tag{6}$$

and

$$\|u_{\lambda,t}\|_{L^2(t_1,t_2,H^{-1}(\mathbb{R}))} \leq M. \tag{7}$$

Q: Three spaces Lemma? $X \hookrightarrow_{comp} B \hookrightarrow Y$, u_λ bounded in $L^2(0,T,X)$ with $u_{\lambda,t}$ bounded in $L^2(0,T,Y)$ imply compactness in $L^2(0,T,B)$

Not exactly ... since we have no gradients :(but an integral term in the second estimate

Compactness

Bourgain, Brezis, Mironescu, Another look to Sobolev Spaces, 2001
Rossi et. al. 2008, A. Ponce 2009

Theorem

Let $1 \leq p < \infty$ *and* $\Omega \subset \mathbb{R}$ *open. Let* $\rho : \mathbb{R} \to \mathbb{R}$ *be a nonnegative smooth continuous radial functions with compact support, non identically zero, and* $\rho_n(x) = n^d \rho(nx)$*. Let* $\{f_n\}$ *be a bounded sequence in* $L^p(\mathbb{R})$ *such that*

$$\int_\Omega \int_\Omega \rho_n(x-y)|f_n(x) - f_n(y)|^p dxdy \leq \frac{M}{n^p}. \tag{8}$$

The following hold:
Assuming that Ω *is a smooth bounded domain in* $\mathbb{R}$ *and* $\rho(x) \geq \rho(y)$ *if* $|x| \leq |y|$ *then* $\{f_n\}$ *is relatively compact in* $L^p(\Omega)$*,.*

Compact sets in $L^p(0,T,B)$

Theorem (Simon '87)

Let $\mathcal{F} \subset L^p(0,T,B)$. $\mathcal{F}$ is relatively compact in $L^p(0,T,B)$ for $1 \leq p < \infty$, or $C(0,T,B)$ for $p = \infty$ if and only if

1. $\{\int_{t_1}^{t_2} f(t)dt,\ f \in \mathcal{F}\}$ *is relatively compact in B for all $0 < t_1 < t_2 < T$.*
2. $\|\tau_h f - f\|_{L^p(0,T-h,B)} \to 0$ *as $h \to 0$ uniformly for $f \in \mathcal{F}$.*

Main idea: put together the previous results to obtain compactness for u_λ

$1 < p < \infty$, Ω smooth, $\rho(x) \geq \rho(y)$ if $|x| \leq |y|$

Theorem

Let $\{f_n\}_{n\geq 1}$ be a sequence of functions in $L^p((0,T)\times\Omega)$ such that

$$\int_0^T \int_\Omega |f_n|^p \leq M, \tag{9}$$

$$n^p \int_0^T \int_\Omega \int_\Omega \rho_n(x-y)|f_n(t,x) - f_n(t,y)|^p dxdydt \leq M. \tag{10}$$

and

$$\|\partial_t f_n\|_{L^p((0,T),W^{-1,p}(\Omega))} \leq M \tag{11}$$

then $\{f_n\}_{n\geq 1}$ is relatively compact in $L^p((0,T)\times\Omega)$.

Proof:

Figure: Caipirinha

Follow carefully the steps in Simon's paper + BBM&R static criterium + tricky inequalities

Consequence: we can apply the "four step method"

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In three space lemma we need something like that

$$X \hookrightarrow_{comp} B \hookrightarrow Y$$

$$\|u\|_B \leq \epsilon \|u\|_X + \eta(\epsilon)\|u\|_Y$$

In the nonlocal setting we have

Lemma

Let $1 < p < \infty$. There exists a positive constant $C = C(\rho, p, d)$ such that for every $\epsilon \in (0,1)$ the following inequality

$$C\|u\|^p_{L^p(\mathbb{R}^d)} \leq \epsilon \Big[n^p \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \rho_n(x-y)|u(x)-u(y)|^p dxdy + \|u\|^p_{L^p(\mathbb{R}^d)} \Big] + \epsilon^{-1}\|u\|^p_{W^{-1,p}(\mathbb{R}^d)}$$

holds for all $n\epsilon^{1/p} \gtrsim 1$ and for all $u \in L^p(\mathbb{R}^d)$.

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holds for all $n \epsilon^{1/p} \gtrsim 1$ and for all $u \in L^p(\mathbb{R}^d)$.

Conclusion: the scaling method works for some nonlocal problems but you have to take care

Future/Related works in the critical case $q = 2$: LEHOUCQ's models, SIAM J. App. Math. 2012

- $u_t = u_{xx} + \int_{\mathbb{R}} K(x-y)(\frac{u(t,x)+u(t,y)}{2})^2 dy$, K odd, $\int K = 0$
- $u_t = J * u - u + \int_{\mathbb{R}} K(x-y)(\frac{u(t,x)+u(t,y)}{2})^2 dy$
- $u_t = u_{xx} + K * u^2 = u_{xx} + \partial_x(G * u^2)$
- In some particular cases there are variants of BBM-B equation

Recently with A. Pazoto, C. Cazacu (see Arxiv) we were able to treat the model

$$u_t = J * u - u - u^{q-1}u_x, 1 < q < 2$$

Obs: $q > 2$ with A. Pazoto, DCDS- 2014

Main idea for $1 < q < 2$: an Oleinik inequality

$$(u^{q-1})_x \leq 1/t \text{ in } \mathcal{D}'(\mathbb{R})$$

even we have a nonlocal operator instead of the Laplacian

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Cordoba & Corboda PNAS 2003

$$(-\Delta)^{\alpha/2}(u^2) - 2u(-\Delta)^{\alpha/2}(u) \leq 0$$

$$Lu = \int_{\mathbb{R}} K(x-y)(u(y)-u(x))dy$$

For any $\beta \geq 0$ and $z \geq 0$ there exists $A_z : \mathbb{R} \to \mathbb{R}$, $A_z \leq 0$ such that

$$\Big(zL(z^\beta w) - \frac{\beta}{\beta+1} wL(z^{\beta+1})\Big)(x_0) \leq A_z(x_0)w(x_0) \tag{12}$$

at the point $x_0 \in \mathbb{R}$ where w attains its maximum.

Obs: $w \equiv 1$, $\beta = 1$ we can take $A_z \equiv 0$.

Obs: $L = u_{xx}$, then $A_z = -\beta z^{\beta-1} z_x^2$

Asymptotic behavior and numerical simulations

L.I. & A. Pozo & E. Zuazua, Math of Comp., 2015

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \qquad x \in \mathbb{R}, t > 0.$$

For large time the solution behavios as a N-wave

$$w_{p,q}(x,t) = \begin{cases} \frac{x}{t}, & -\sqrt{2pt} < x < \sqrt{2qt}, \\ 0, & \text{elsewhere.} \end{cases} \tag{13}$$

Lor the Lax-Friedrichs scheme, $w = w_{M_\Delta}$ is the unique solution of the continuous viscous Burgers equation

$$\begin{cases} w_t + \left(\frac{w^2}{2}\right)_x = \frac{(\Delta x)^2}{2} w_{xx}, \quad x \in \mathbb{R}, t > 0, \\ w(0) = M_\Delta \delta_0, \end{cases} \tag{14}$$

with $M_\Delta = \int_{\mathbb{R}} u_\Delta^0$.

w - parabolic profile

For Engquist-Osher and Godunov schemes, $w = w_{p_\Delta, q_\Delta}$ is the unique solution of the hyperbolic Burgers equation

$$\begin{cases} w_t + \left(\frac{w^2}{2}\right)_x = 0, \quad x \in \mathbb{R}, t > 0, \\ w(0) = M_\Delta \delta_0, \quad \lim_{t \to 0} \int_{-\infty}^{x} w(t,z)dz = \begin{cases} 0, & x < 0, \\ -p_\Delta, & x = 0, \\ q_\Delta - p_\Delta, & x > 0, \end{cases} \end{cases} \tag{15}$$

with $M_\Delta = \int_{\mathbb{R}} u_\Delta^0$ and

$$p_\Delta = -\min_{x \in \mathbb{R}} \int_{-\infty}^{x} u_\Delta^0(z)dz \quad \text{and} \quad q_\Delta = \max_{x \in \mathbb{R}} \int_{x}^{\infty} u_\Delta^0(z)dz.$$

w - hyperbolic profile

Some "more or less" Open Problems

- Second term for the above models
- $u_t = u_{xx} + G * u^q - u^q$, $1 < q < 2$
- $u_t = J * u - u + G * u^q - u^q$, $1 < q < 2$
- Joint work with D. Stan, $s \in (1,2)$
  $u_t + (-\Delta)^{s/2} u + (u^q)_x = 0$, $1 < q < q_s < 2$
- Asymptotic expansion for the solutions of the numerical approximations of subcritical convection-diffusion
- Other models where the convection dominates the diffusion
- Joint work with J. Rossi, nonlocal Oleinik estimates
- Understand the competition between diffusion and the nonlocal convection

THANKS for your attention !!!