Quasi-geostrophic equation in $\mathbb{R}^2$

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Abstract

Solvability of Cauchy's problem in $\mathbb{R}^2$ for subcritical quasi-geostrophic equation is discussed here in phase spaces $H^s(\mathbb{R}^2)$ with $s > 1$. A solution to that equation in critical case is obtained next as a limit of the H^s-solutions to subcritical equations when the exponent α of $(-\Delta)^\alpha$ tends to $\frac{1}{2}^+$. Such idea seems to be new in the literature.

1. Introduction

The dissipative quasi-geostrophic equation considered here has the form:

$$\begin{aligned} &\theta_t + u\cdot\nabla\theta + \kappa(-\Delta)^\alpha\theta = f, \quad x\in\mathbb{R}^2,\ t>0,\\ &\theta(0,x) = \theta_0(x), \end{aligned} \tag{1.1}$$

where θ represents the potential temperature, $\kappa > 0$ is a diffusivity coefficient, $\alpha \in [\frac{1}{2}, 1]$ a fractional exponent, and $u = (u_1, u_2)$ is the *velocity field* determined by θ through the relation:

$$u = \left(-\frac{\partial\psi}{\partial x_2}, \frac{\partial\psi}{\partial x_1}\right), \quad \text{where} \quad (-\Delta)^{\frac{1}{2}}\psi = -\theta, \tag{1.2}$$

or, in a more explicit way,

$$u = \left(-R_2\theta, R_1\theta\right), \tag{1.3}$$

where $R_i, i = 1, 2$ are the *Riesz transforms.*

Description of the results. We study the global in time solvability and properties of solutions to the Cauchy problem (1.1). We chose $H^s(\mathbb{R}^2)$ with $s > 1$ as the *base spaces*. Our aim was to include, in the subcritical case of exponent $\alpha \in (\frac{1}{2}, 1]$, the problem (1.1) into the frame of semilinear parabolic equations with sectorial operator (see [4, 1]). This offers a simple but formalized proof of the local solvability and regularity in the subcritical case. The presented approach to critical problem (1.1) with $\alpha = \frac{1}{2}$ is new here. However, using weak compactness of bounded sets in reflexive Banach space as a tool for getting convergence to a weak solution of the critical problem, we will not be able to show that the limit of the nonlinearities of subcritical problems equals the nonlinearity of the limiting critical problem. The existing uniform in $\alpha \in (\frac{1}{2}, 1]$ a priori estimates are too weak to guarantee such property. However, they work well in case of all the linear components in the equation. The main result obtained in that direction is formulated in Theorem 3.2, where we introduce the notion of the *weak L^p solution* to the critical equation (1.1).

2. Formulation of the problem and its local solvability

Our first task is the local in time solvability of the subcritical problem (1.1) when the equation is treated in the *base space* $X := H^s(\mathbb{R}^2)$, $s > 1$. The resulting phase space is $H^{2\alpha^- + s}(\mathbb{R}^2)$. We use an approach of Dan Henry [4]. To work with a *sectorial positive* operator, we rewrite (1.1) in an form:

$$\begin{aligned} &\theta_t + u\cdot\nabla\theta + \kappa(-\Delta)^\alpha\theta + \kappa\theta = f + \kappa\theta, \quad x\in\mathbb{R}^2,\ t>0,\\ &\theta(0,x) = \theta_0(x). \end{aligned} \tag{2.1}$$

Define $A_\alpha := \kappa[(-\Delta)^\alpha + I]$, $\alpha \in (\frac{1}{2}, 1]$, where $(-\Delta)^\alpha$ is the fractional Laplacian. Also, setting

$$F(\theta) = R_2\theta\frac{\partial\theta}{\partial x_1} - R_1\theta\frac{\partial\theta}{\partial x_2} + f + \kappa\theta, \tag{2.2}$$

the problem (2.1) will be written formally as

$$\begin{aligned} &\theta_t + A_\alpha\theta = F(\theta),\ t>0,\\ &\theta(0) = \theta_0, \end{aligned} \tag{2.3}$$

which is an abstract 'parabolic' equation with sectorial positive operator.

Theorem 2.1. *Let $s > 1$ be fixed. Then, for $f \in H^s(\mathbb{R}^2)$ and for arbitrary $\theta_0 \in H^{2\alpha^-+s}(\mathbb{R}^2)$, there exists in the phase space $H^{2\alpha^-+s}(\mathbb{R}^2)$ a unique local in time mild solution $\theta(t)$ to the subcritical problem* (1.1)*, $\alpha \in (\frac{1}{2}, 1]$. Moreover,*

$$\theta \in C((0,\tau); H^{2\alpha+s}(\mathbb{R}^2)) \cap C([0,\tau); H^{2\alpha^-+s}(\mathbb{R}^2)),\ \theta_t \in C((0,\tau); H^{2\gamma+s}(\mathbb{R}^2)),$$

with arbitrary $\gamma < \alpha^-$. Here $\tau > 0$ is the 'life time' of that local solution. Moreover, the Cauchy formula *is satisfied:*

$$\theta(t) = e^{-A_\alpha t}\theta_0 + \int_0^t e^{-A_\alpha(t-s)}F(\theta(s))ds,\ t\in[0,\tau),$$

where $e^{-A_\alpha t}$ denotes the linear semigroup corresponding to the operator $A_\alpha := \kappa[(-\Delta)^\alpha + I]$ in $H^s(\mathbb{R}^2)$, and F is given by formula (2.2)*.*

Proof. We need to check that the nonlinearity (2.2) is Lipschitz continuous on bounded sets as a map from $H^{2\alpha^-+s}(\mathbb{R}^2)$ into $H^s(\mathbb{R}^2)$, $s > 1$. We have

$$\begin{aligned} &\|F(\theta_1) - F(\theta_2)\|_{H^s(\mathbb{R}^2)} \le \|R_2(\theta_1-\theta_2)\frac{\partial\theta_1}{\partial x_1} + R_2\theta_2\frac{\partial(\theta_1-\theta_2)}{\partial x_1}\|_{H^s(\mathbb{R}^2)}\\ &+ \|R_1(\theta_1-\theta_2)\frac{\partial\theta_1}{\partial x_2} + R_1\theta_2\frac{\partial(\theta_1-\theta_2)}{\partial x_2}\|_{H^s(\mathbb{R}^2)} + \kappa\|\theta_1-\theta_2\|_{H^s(\mathbb{R}^2)}. \end{aligned} \tag{2.4}$$

Since $H^s(\mathbb{R}^2)$ is a *Banach algebra* we have

$$\|R_2(\theta_1-\theta_2)\frac{\partial\theta_1}{\partial x_1}\|_{H^s(\mathbb{R}^2)} \le\le c\||u_1-u_2|\|_{H^s(\mathbb{R}^2)}\|\theta_1\|_{H^{2\alpha^-+s}(\mathbb{R}^2)}. \tag{2.5}$$

Applying the property (4.3) to the first term in (2.5) we get

$$\|R_2(\theta_1-\theta_2)\frac{\partial\theta_1}{\partial x_1}\|_{H^s(\mathbb{R}^2)} \le c\|\theta_1-\theta_2\|_{H^{2\alpha^-+s}(\mathbb{R}^2)}\|\theta_1\|_{H^{2\alpha^-+s}(\mathbb{R}^2)}.$$

The others components in (2.4) are estimated analogously. □

Global solvability. To guarantee the global in time solvability of (1.1) in $H^{2\alpha^-+s}(\mathbb{R}^2)$ the a priori estimate (2.7) below will be used.

The Maximum Principle

Lemma 2.2. *Let $q \in [2,\infty)$ and $f \in L^q(\mathbb{R}^2)$. Then, for a sufficiently regular solution of* (1.1)*, the following estimate holds:*

$$\|\theta(t,\cdot)\|^q_{L^q(\mathbb{R}^2)} \le \|\theta_0\|^q_{L^q(\mathbb{R}^2)}e^{(q-1)t} + \frac{e^{(q-1)t}-1}{q-1}\|f\|^q_{L^q(\mathbb{R}^2)}. \tag{2.6}$$

Proof. Multiplying (1.1) by $|\theta|^{q-1}sgn(\theta)$, we obtain

$$\int_{\mathbb{R}^2}\theta_t|\theta|^{q-1}sgn(\theta)dx + \kappa\int_{\mathbb{R}^2}(-\Delta)^\alpha\theta|\theta|^{q-1}sgn(\theta)dx = \int_{\mathbb{R}^2}f|\theta|^{q-1}sgn(\theta)dx.$$

Since for all the values $1 \le q < \infty$ and $0 \le \alpha \le 1$, as was shown in [2],

$$\int_{\mathbb{R}^2}(-\Delta)^\alpha\theta|\theta|^{q-1}sgn(\theta)dx \ge 0,$$

using Hölder and Young inequalities, we get

$$\frac{1}{q}\frac{d}{dt}\int_{\mathbb{R}^2}|\theta|^q dx \le \int_{\mathbb{R}^2}f|\theta|^{q-1}sgn(\theta)dx \le \frac{1}{q}\|f\|^q_{L^q(\mathbb{R}^2)} + \frac{q-1}{q}\|\theta\|^q_{L^q(\mathbb{R}^2)}.$$

□

Further a priori estimates. Following the calculations in [13, p. 1165] we are able to estimate higher Sobolev norms of the solutions to (1.1). Let $l \ge \alpha$ be fixed and $f \in H^{l-\alpha}(\mathbb{R}^2)$. Multiplying the equation by $(-\Delta)^l\theta$ we obtain a differential inequality:

$$\frac{d}{dt}\int_{\mathbb{R}^2}[(-\Delta)^{\frac{l}{2}}\theta]^2dx + \kappa\int_{\mathbb{R}^2}[(-\Delta)^{\frac{l+\alpha}{2}}\theta]^2dx \le c(\|\theta_0\|_{L^\infty(\mathbb{R}^2)}, \|\theta_0\|_{L^2(\mathbb{R}^2)}; \|f\|_{H^{l-\alpha}(\mathbb{R}^2)}). \tag{2.7}$$

3. Critical equation (1.1); $\alpha = \frac{1}{2}$.

Passing to the limit. Letting $\alpha \to \frac{1}{2}^+$ in the equation (1.1) is given next. We consider the solutions θ^α. We add the superscript for clarity. We look at (2.1) as an equation in $L^p(\mathbb{R}^2)$, $p \ge 2$, and 'multiply' it by the test function $(A_\alpha^{-1})^*\phi$ where $\phi \in \mathcal{L}^q_{-2}(\mathbb{R}^2)$, $\frac{1}{p} + \frac{1}{q} = 1$, to get:

$$\begin{aligned} &< [\theta_t^\alpha + u\cdot\nabla\theta^\alpha], (A_\alpha^{-1})^*\phi >_{L^p,L^q} = -< A_\alpha\theta^\alpha, (A_\alpha^{-1})^*\phi >_{L^p,L^q}\\ &+ < [f + \kappa\theta^\alpha], (A_\alpha^{-1})^*\phi >_{L^p,L^q}. \end{aligned} \tag{3.1}$$

We will discuss now the convergence of the terms in (3.1) one by one. Note that when $\alpha \to \frac{1}{2}^+$, then by Lemma 4.2, $(A_\alpha^{-1})^*\phi \to (A_{\frac{1}{2}}^{-1})^*\phi$ for $\phi \in \mathcal{L}^q_{-2}(\mathbb{R}^2)$. Thanks to boundedness of θ^α in $L^p(\mathbb{R}^2)$, $p \ge 2$, uniform in $\alpha \in (\frac{1}{2}, 1]$ (Lemma 2.2), adding and subtracting, we obtain:

$$\begin{aligned} &< [f + \kappa\theta^\alpha], (A_\alpha^{-1})^*\phi >_{L^p,L^q} = < [f + \kappa\theta^\alpha], (A_\alpha^{-1})^*\phi - (A_{\frac{1}{2}}^{-1})^*\phi >_{L^p,L^q}\\ &+ < [f + \kappa\theta^\alpha], (A_{\frac{1}{2}}^{-1})^*\phi >_{L^p,L^q} \to < [f + \kappa\theta], (A_{\frac{1}{2}}^{-1})^*\phi >_{L^p,L^q}, \end{aligned} \tag{3.2}$$

where θ is the weak limit of θ^α in $L^p(\mathbb{R}^2)$ as $\alpha \to \frac{1}{2}^+$ (over a sequence $\{\alpha_n\}$ convergent to $\frac{1}{2}^+$; various sequences may lead to various weak limits).
For the intermediate term, we have the equality:

$$< A_\alpha\theta^\alpha, (A_\alpha^{-1})^*\phi >_{L^p,L^q} = < \theta^\alpha, \phi >_{L^p,L^q}. \tag{3.3}$$

Moreover, since $\theta^\alpha \in W^{1,p}(\mathbb{R}^2)$, $p \ge 2$, we have: $\theta^\alpha \in L^p(\mathbb{R}^2)$ and $A_{\frac{1}{2}}\theta^\alpha =: T^\alpha \in L^p(\mathbb{R}^2)$, which gives that $\theta^\alpha = A_{\frac{1}{2}}^{-1}T^\alpha$ for some $T^\alpha \in L^p(\mathbb{R}^2)$. Inserting to the equation above we obtain:

$$< \theta^\alpha, \phi >_{L^p,L^q} = < A_{\frac{1}{2}}^{-1}T^\alpha, \phi >_{L^p,L^q} = < T^\alpha, (A_{\frac{1}{2}}^{-1})^*\phi >_{L^p,L^q}.$$

Since the left hand side has a limit as $\alpha_n \to \frac{1}{2}^+$, the right hand side also has the same limit:

$$< T^\alpha, (A_{\frac{1}{2}}^{-1})^*\phi >_{L^p,L^q} \to < \Theta, (A_{\frac{1}{2}}^{-1})^*\phi >_{L^p,L^q} = < \theta, \phi >_{L^p,L^q} \ \text{for all } \phi \in \mathcal{L}^q_{-2}(\mathbb{R}^2);$$

note that $\mathcal{L}^q_{-2}(\mathbb{R}^2)$ is dense in $L^q(\mathbb{R}^2)$ and that $\theta, \Theta \in L^p(\mathbb{R}^2)$.
Consequently, returning to (3.1), we see that the first term also has a limit:

$$< [\theta_t^\alpha + u\cdot\nabla\theta^\alpha], (A_\alpha^{-1})^*\phi >_{L^p,L^q} \to \omega_\phi, \tag{3.4}$$

as $\alpha \to \frac{1}{2}^+$. Passing countable many times to a subsequence, all the limits above can be achieved on a dense subset of the *separable* space $L^q(\mathbb{R}^2)$. Thus we find a weak form of the limit equation:

$$\omega_\phi = -< \theta, \phi >_{L^p,L^q} + < f + \kappa\theta, (A_{\frac{1}{2}}^{-1})^*\phi >_{L^p,L^q} \quad \text{for all } \phi \in \mathcal{L}^q_{-2}(\mathbb{R}^2). \tag{3.5}$$

Remembering that the set $\{(A_{\frac{1}{2}}^{-1})^*\phi; \phi \in \mathcal{L}^q_{-2}(\mathbb{R}^2)\}$ is dense in $L^q(\mathbb{R}^2)$, the right hand side above defines a unique element in $L^p(\mathbb{R}^2)$.

Separation of terms. The time derivative will be separated from the term $[\theta_t^\alpha + u\cdot\nabla\theta^\alpha]$ when letting $\alpha \to \frac{1}{2}^+$. More precisely we formulate:

Remark 3.1. *Since the approximating solutions θ^α satisfy*

$$\theta^\alpha \in L^\infty(0,T; L^p(\mathbb{R}^2)), \theta_t^\alpha \in L^2(0,T; L^p(\mathbb{R}^2)),\ \alpha \in (\frac{1}{2}, \frac{3}{4}], \tag{3.6}$$

then by [11, Lemma 1.1, Chapt.III]

$$< \theta_t^\alpha, \eta >_{L^p,L^q} = \frac{d}{dt} < \theta^\alpha, \eta >_{L^p,L^q} \to \frac{d}{dt} < \theta, \eta >_{L^p,L^q} \quad \text{for all } \eta \in L^q(\mathbb{R}^2), \tag{3.7}$$

(here $\frac{1}{p} + \frac{1}{q} = 1$) the derivative $\frac{d}{dt}$ and the convergence are in $\mathcal{D}'(0,T)$ (space of the 'scalar distributions'). Consequently,

$$\omega_\phi = \frac{d}{dt} < \theta, (A_{\frac{1}{2}}^{-1})^*\phi >_{L^p,L^q} + \omega^1_\phi, \tag{3.8}$$

where ω^1_ϕ is a limit in $\mathcal{D}'(0,T)$ of $< u\cdot\nabla\theta^\alpha, (A_\alpha^{-1})^*\phi >_{L^p,L^q}$ over a chosen sequence $\alpha_n \to \frac{1}{2}^+$.

The construction presented above allows us to formulate the following theorem:

Theorem 3.2. *Let $\{\theta^\alpha\}_{\alpha\in(\frac{1}{2},\frac{3}{4}]}$ be the set of regular $H^{s+2\alpha^-}(\mathbb{R}^2)$ solutions to subcritical equation* (1.1)*. Such solutions are, in particular, bounded in each space $L^p(\mathbb{R}^2)$ for $p \in [2,+\infty)$, uniformly in α. As a consequence of that and the smoothness properties of regular solutions (they vary continuously in $W^{1,p}(\mathbb{R}^2)$), for arbitrary sequence $\{\alpha_n\} \subset (\frac{1}{2}, \frac{3}{4}]$ convergent to $\frac{1}{2}$ we can find a subsequence $\{\alpha_{n_k}\}$ that the corresponding sequence $\{\theta^{n_k}\}$ converges weakly in $L^p(\mathbb{R}^2)$ to a function θ fulfilling the equation:*

$$\omega_\phi = -< \theta, \phi >_{L^p,L^q} + < f + \kappa\theta, (A_{\frac{1}{2}}^{-1})^*\phi >_{L^p,L^q} \quad \text{for all } \eta \in L^q(\mathbb{R}^2). \tag{3.9}$$

Due to denseness of the set $\mathcal{L}^q_{-2}(\mathbb{R}^2)$ (see Definition 4.3) in $L^q(\mathbb{R}^2)$, $\frac{1}{p} + \frac{1}{q} = 1$, the right hand side of (3.9) *defines a unique element in $L^p(\mathbb{R}^2)$. The left hand side ω_ϕ is defined in* (3.4) *and discussed in Remark 3.1. We will call such θ a* weak L^p solution *to the critical equation* (1.1)*, $\alpha = \frac{1}{2}$.*

4. Appendix.

Lemma 4.1. *When $(\frac{1}{2}, 1] \ni \alpha \to \frac{1}{2}^+$, then $A^{-\alpha} \to A^{-\frac{1}{2}}$ pointwise on the domain of A^{-1}, where A is a sectorial non-negative operator in a reflexive Banach space X.*

Lemma 4.2. *Let $p \in [2,+\infty)$ be arbitrary. When $(\frac{1}{2}, 1] \ni \alpha \to \frac{1}{2}^+$, then $A_\alpha^{-1} := \frac{1}{\kappa}[(-\Delta)^\alpha + I]^{-1} \to A_{\frac{1}{2}}^{-1}$ pointwise in*

$$D((-\Delta)^{-1}) = \{\phi \in L^p(\mathbb{R}^2); (-\Delta)^{-1}\phi \in L^p(\mathbb{R}^2)\} = \mathcal{L}^p_{-2}(\mathbb{R}^2);$$

compare (4.1) *for the definition of $\mathcal{L}^p_{-2}(\mathbb{R}^2)$.*

Definition 4.3. *For $1 \le p \le \infty$ and $\gamma \in \mathbb{R}$ define*

$$\mathcal{L}^p_\gamma(\mathbb{R}^N) = \{f \in L^p(\mathbb{R}^N) \colon f = I^\gamma g := (-\Delta)^{-\frac{\gamma}{2}}g \text{ for certain } g \in L^p(\mathbb{R}^N)\}, \tag{4.1}$$

normed by $\|f\|_{\mathcal{L}^p_\gamma(\mathbb{R}^N)} = \|g\|_{L^p(\mathbb{R}^N)}$.

The *commutator estimate* is known in the literature: (see[8, p. 61]) for $\gamma > 0$,

$$\|(-\Delta)^\gamma(FG)\|_{L^p(\mathbb{R}^2)} \le c\big[\|(-\Delta)^\gamma F\|_{L^r(\mathbb{R}^2)}\|G\|_{L^q(\mathbb{R}^2)} + \|F\|_{L^q(\mathbb{R}^2)}\|(-\Delta)^\gamma G\|_{L^r(\mathbb{R}^2)}\big],$$

where $1 < p < q \le \infty$ and $\frac{1}{r} + \frac{1}{q} = \frac{1}{p}$, and the right hand side is sensible for F and G.
Moreover, we quote the observation (e.g. [10, Chapter III], [6, p. 299]), that *the Riesz transforms R_j are bounded operators from $L^q(\mathbb{R}^N)$ into $L^q(\mathbb{R}^N)$,* $1 < q < \infty$:

$$\exists_{C>0}\forall_{\psi\in L^q(\mathbb{R}^N)}\|R_j(\psi)\|_{L^q(\mathbb{R}^N)} \le C\|\psi\|_{L^q(\mathbb{R}^N)}, \quad j = 1, 2, ..., N. \tag{4.2}$$

We have also the following property taken from the paper [12, p. 12]:

$$\||D^j u(t,\cdot)|\|_{L^q(\mathbb{R}^2)} \le c\|D^j\theta(t,\cdot)\|_{L^q(\mathbb{R}^2)},\ q \in (1,\infty),\ |j| \le k. \tag{4.3}$$

Properties of the fractional powers operators. Recall first the Balakrishnan definition of fractional power of non-negative operator (e.g. [5, p. 299]). Let A be a closed linear densely defined operator in a Banach space X, such that its resolvent set contains $(-\infty, 0)$ and the resolvent satisfies:

$$\|\lambda(\lambda + A)^{-1}\| \le M,\ \lambda > 0.$$

Then, for $\eta \in (0,1)$,

$$A^\eta\phi = \frac{\sin(\pi\eta)}{\pi}\int_0^\infty s^{\eta-1}A(s+A)^{-1}\phi ds.$$

Note that there is another definition, through singular integrals, of the fractional powers of the $(-\Delta)^{-\alpha}$ in $L^p(\mathbb{R}^N)$ frequently used in the literature. See [6, Chapter 2.2] for the proof of equivalence of the two definitions for $1 < p < \frac{N}{2\Re\alpha}$; see also [3, section 4.3].

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