Porous Medium Flow with a Fractional Potential Pressure and Fractional Time Derivative

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joint work with L. Caffarelli and A. Vasseur

1. Porous Medium Equation
   - Diffusion
2. Fractional Diffusion
   - Fractional Laplacian
3. Fractional Time Derivative
   - Caputo or Marchaud Derivative
4. Results
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Porous Medium Equation

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In nondivergence form

$$u_t - \Delta u^m = 0.$$

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$$\partial\{u(x, t) > 0\}$$

is considered the free boundary.

Fractional Laplacian

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Note: $(-\Delta)^{-s}$ is the Reisz potential

$$(-\Delta)^{-s} f(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2s}} dy.$$

A fractional potential pressure

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- Solutions also exhibit finite time propagation.

Fractional Derivatives

Riemann-Liouville derivative for $n-1<\alpha<n$

$$ {}_aD_t^{\alpha} f(t) = \frac{d^n}{dt^n}\frac{1}{\Gamma(n-\alpha)}\int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\, d\tau. $$

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Note: The two operators are identical if $a=-\infty$.

Various Formulations

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Letting $f(t) = f(a)$ for $t < a$ we have

$$ \Gamma(1-\alpha)\ {}_a^cD_t^\alpha f(t) = \alpha \int_{-\infty}^t \frac{f(t)-f(s)}{(t-s)^{1+\alpha}}\ ds, $$

or the Marchaud derivative.

An Energy term in time

$$
\begin{array}{ccccc}
c_\alpha \int_a^T 2uD_t^\alpha u = & \int_a^T \int_a^t \frac{[u(t)-u(s)]^2}{(t-s)^{1+\alpha}} & + & \int_a^T \frac{u^2(t)}{(T-t)^\alpha} - \frac{u^2(t)}{(t-a)^\alpha} \\
\downarrow \quad = & \downarrow & & \downarrow \\
\int_a^T 2u\partial_t u = & 0 & + & u^2(T) - u^2(a).
\end{array}
$$

This useful term in energy disappears as $\alpha \to 1$.

A Generalized Fractional Time Derivative

For energy methods one may consider the generalized derivative

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Similar to divergence form elliptic equations with measurable coefficients one only needs

$$\frac{\Lambda^{-1}}{(t-s)^{1+\alpha}} \leq K(t,s,x) \leq \frac{\Lambda}{(t-s)^{1+\alpha}}$$

with the relation

$$K(t, t-s) = K(t+s, t).$$

Porous Medium Flow with a fractional time derivative

When the permeability of the medium changes over time one may consider an equation with "memory".

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We consider weak solutions to the equation

$$ \mathcal{D}_t^\alpha u - \text{div}(u\nabla(-\Delta)^{-s}u) = f $$

Weak formulation

$$\int \phi(-\Delta)^s \psi = c \int \int \frac{[\phi(x) - \phi(y)][\psi(x) - \psi(y)]}{|x-y|^{n+2s}}$$

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$$\begin{aligned}\int_{-\infty}^{T} \phi D_t^\alpha \psi \ dt = c \int_{-\infty}^{T} \int_{-\infty}^{t} \frac{[\phi(t)-\phi(s)][\psi(t)-\psi(s)]}{(t-s)^{1+\alpha}} \ ds \ dt \\ + c \int_{-\infty}^{T} \frac{\phi(t)\psi(t)}{(T-t)^\alpha} \ dt \\ - \int_{-\infty}^{T} \psi D_t^\alpha \phi(t) \ dt.\end{aligned}$$

Existence

Theorem (A., Caffarelli, Vasseur '16)

Let $0 < s < 1/2$ *and* $n > 1$. *Let* $0 \leq u_0, f(x) \leq Ae^{-|x|}$ *and assume further that* $u_0 \in C^2$. *Then there exists a weak solution on* $\mathbb{R}^n \times (0, \infty)$ *with initial data* u_0.

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Theorem (A., Caffarelli, Vasseur '16)

Let $0 < s < 1/2$ and $n > 1$. Let u be the weak solution obtained above. Then u is Hölder continuous on $\mathbb{R}^n \times [0, T]$ for all $0 < T < \infty$.

Method

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Lemma

Under the appropriate assumptions (including $0 \leq u \leq 1$), if

$$|\{u < 1/2\} \cap Q_2| \geq \kappa |Q_2|,$$

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If at some point the above Lemma does not hold, then...

Second Lemma

Lemma

Under the appropriate assumptions (including $0 \leq u \leq 1$), if

$$|\{u \geq 1/2\} \cap Q_2| \geq (1-\kappa)|Q_2|,$$

then $u \geq \mu$ on Q_1.

In this situation the equation is no longer degenerate and the usual De Giorgi method applies.

Approximate Problems

To prove existence we first prove existence of the approximate solutions

$$\mathcal{D}_\epsilon^\alpha u - \delta \text{div}((d_1 + d_2 u)\nabla u) - \text{div}((d_1 + d_2 u)\nabla K_\zeta u) = f.$$

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We actually prove the Lemmas for the approximate solutions.

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- Introducing a right hand side allows us to prove Hölder continuity up to the intial data.
- A right hand side is a stronger result.

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Our test function is

$$-\left[\left(\frac{u}{\psi}\right)^\gamma - 1\right]_- .$$

Integration by parts

$$2u(t)[u(t) - u(s)] = [u(t) - u(s)]^2 + u^2(t) - u^2(s)$$

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$$\int_{-\infty}^{T} \int_{\infty}^{t} [u^2(t) - u^2(s)] K(t,s)\, ds\, dt = \int_{-\infty}^{T} \int_{-\infty}^{2t-T} u^2(t) K(t,s)\, ds\, dt.$$

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Then

$$\int_{-\infty}^{T} u(t)\mathcal{D}_t^{\alpha} u \approx \|u\|_{W^{\alpha/2,2}}$$

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If F is convex with $F'' \geq 2b$, then

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Notice that the anti-derivative of x^γ is convex for $\gamma > 0$.

Fractional Sobolev space

By utilizing u^γ rather than $\log u$ our fractional Sobolev space for the spatial variable is

$$W^{\frac{2-2s}{2+\gamma},2+\gamma},$$

rather than the Hilbet space

$$W^{1-s,2}.$$

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- Prove existence and regularity for $s \in [1/2, 1)$.
- Prove uniform estimates for $s \to 1$ and $\alpha \to 1$.
- Relax conditions on right hand side f and initial data u_0.
- Study the free boundary $\partial\{u(x,t) > 0\}$.