An equivalence between the Dirichlet and the Neumann problem for the Laplace operator

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Abstract

The Neumann problem is in general "*harder*" than the Dirichlet problem. In this talk we show in certain cases they are "*equally hard*"/equivalent, in the sense that solving one of them leads to the solution of the other one.

More precisely, we give a representation of the solution of the Neumann problem for the Laplace operator on the unit ball in $\mathbb{R}^n$ ($n \geq 1$) in terms of the solution of an associated Dirichlet problem.

The representation is suitable for extensions, and we provide extensions to:

a) other operators besides the Laplacian
b) smooth planar domains
c) infinite dimensional case
d) general boundary data.

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Preliminaries

Consider the Dirichlet and Neumann problems for the Laplacian on a smooth bounded domain $D \subset \mathbb{R}^n$:

$$\begin{cases} \Delta u = 0 \text{ in } D \\ u = \varphi \text{ on } \partial D \end{cases}, \tag{1}$$

respectively

$$\begin{cases} \Delta U = 0 \text{ in } D \\ \frac{\partial U}{\partial \nu} = \phi \text{ on } \partial D \end{cases}, \tag{2}$$

where ν is the outward unit normal to the boundary of D, and $\varphi, \phi : \partial D \to \mathbb{R}$ are the given boundary values.

In the case $D = \mathbb{U} = \{z \in \mathbb{R}^n : |z| < 1\}$ the Dirichlet problem (1) has a unique solution for continuous boundary data φ on $\partial \mathbb{U}$.

The Neumann problem (2) also has a solution, unique up to additive constants, for continuous boundary data ϕ on $\partial \mathbb{U}$ satisfying the centering hypothesis $\int_{\partial D} \phi(z)\, \sigma(dz) = 0$.

The centering condition is a necessary condition for the existence of a solution, since by Green's first identity we have

$$\int_{\partial D} \phi(z)\, \sigma(dz) = \int_{\partial D} 1 \frac{\partial U}{\partial \nu}(z)\, \sigma(dz) = \int_D 1 \Delta U(z) + \nabla 1 \cdot \nabla U(z)\, dz = 0.$$

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There are several representations of the solutions of the Dirichlet and Neumann problems above in the literature:

- by single / double layer potentials
- by spherical harmonics
- by probabilistic methods

The probabilistic representation of the solution of the Dirichlet problem (1) is

$$u(x) = E^x \varphi(B_{\tau_D}), \tag{3}$$

where $(B_t)_{t\geq 0}$ is a n-dimensional Brownian motion starting at $B_0 = x$ and $\tau_D = \inf\{t \geq 0 : B_t \notin D\}$ denotes its lifetime.
The probabilistic representation of the Neumann problem (2) is given by

$$U(x) = \lim_{t\to\infty} \frac{1}{2} E^x \int_0^t \phi(X_s) dL_s, \tag{4}$$

where $(X_t)_{t\geq 0}$ is reflecting Brownian motion in D starting at $X_0 = x$ and $(L_t)_{t\geq 0}$ is the boundary local time of X.

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Remark 1

One of our results (Theorem 1) shows that in the case of the unit ball we can find the solution of the Neumann problem by solving the Dirichlet problem ***with the same boundary values****.*

Probabilistically, this result has an interesting consequence, since the solution of the Neumann problem is related to the reflecting Brownian motion, while the Dirichlet problem is related to the (killed) Brownian motion.

The result in Theorem 1 establishes thus a connection between reflecting and killed/free Brownian motion, a result of independent interest.

Notation. We identify the complex plane $\mathbb{C}$ with $\mathbb{R}^2$.
In particular, the dot product of two vectors $a = (a_1, a_2)$, $b = (b_1, b_2) \in \mathbb{R}^2$ will be written

$$a \cdot b = a_1 b_1 + a_2 b_2 = \mathrm{Re}\left(a\bar{b}\right),$$

where on the right we identify a and b with the corresponding complex numbers.
Also, for a function u defined on a subset of $\mathbb{R}^2$ (or $\mathbb{C}$), we will write equivalently $u(x, y)$ or $u(z)$, where $z = x + iy \in D$.

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Heuristics (the case of the unit ball $\mathbb{U} \subset \mathbb{R}^2$)

Consider $D = \mathbb{U} = \left\{ z \in \mathbb{R}^2 : \|z\| < 1 \right\}$ the unit disk.

If u is a solution to the Dirichlet problem (1), then u is the real part of an analytic function $G = u + iv$ in $\mathbb{U}$ (v can be determined from the Cauchy-Riemann equations for G).

The Dirichlet problem (1) is thus equivalent to the problem of finding an analytic function in $\mathbb{U}$, continuous up to the boundary, with prescribed boundary values φ for its real part.

Similarly, if U is a solution of the Neumann problem (2), then U is the real part of an analytic function $F = U + iV$. Since the outward unit normal to $\partial\mathbb{U}$ is $\nu(z) = z$, $z \in \partial\mathbb{U}$, the Neumann boundary condition for U can be written

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If we set $G(z) = zF'(z)$, $z \in \mathbb{U}$, it follows that G is analytic in $\mathbb{U}$ and has boundary values ϕ for its real part. Once G is determined like this, we can find F by complex integration

$$F(z) = F(0) + \int_0^z \frac{G(\xi)}{\xi} d\xi, \qquad z \in \mathbb{U}, \tag{5}$$

and we can then determine the solution of the Neumann problem (2) as $U(z) = \mathrm{Re} F(z)$.
The Neumann problem (2) is therefore equivalent to the problem of finding the analytic function G in $\mathbb{U}$, continuous up to the boundary, with prescribed values ϕ for its real part.
The above heuristics show that both the Dirichlet and Neumann problems (at least in the case of the unit disk) are equivalent to finding an analytic function in $\mathbb{U}$, continuous up to the boundary, with prescribed values on the boundary for its real part. So the Dirichlet and Neumann problems are "equally hard" in this case.
Formula (5) suggests a direct way of finding the solution $U = \mathrm{Re} F$ of the Neumann problem from the solution $u = \mathrm{Re} G$ of the Dirichlet problem, by circumventing the problem of finding the corresponding analytic functions F and G.
Integrating in (5) along the line segment from 0 to $z \in \mathbb{U}$, we obtain

$$F(z) = F(0) + \int_0^1 \frac{G(\rho z)}{\rho} d\rho,$$

and taking real parts we obtain

$$U(z) = \mathrm{Re} F(0) + \int_0^1 \frac{u(\rho z)}{\rho} d\rho, \qquad z \in \mathbb{U}.$$

If we set $G(z) = zF'(z)$, $z \in \mathbb{U}$, it follows that G is analytic in $\mathbb{U}$ and has boundary values ϕ for its real part. Once G is determined like this, we can find F by complex integration

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The case of the unit ball $\mathbb{U} \subset \mathbb{R}^n$: the main result

Theorem 1 ([BePaPa])

Assume $\phi : \partial\mathbb{U} \to \mathbb{R}$ is continuous and satisfies $\int_{\partial\mathbb{U}} \phi(z)\,\sigma(dz) = 0$. If u is the solution of the Dirichlet problem (1) with boundary condition $\varphi = \phi$ on $\partial\mathbb{U}$, then

$$U(z) = \int_0^1 \frac{u(\rho z)}{\rho} d\rho, \qquad z \in \mathbb{U}, \tag{6}$$

is the solution to the Neumann problem (2) with $U(0) = 0$.

Proof. The previous heuristics could serve as a proof in the particular case $n = 2$, provided we show that the analytic functions F and G can be extended to the boundary of $\mathbb{U}$.
Instead, a direct proof can be worked out in the general case $n \geq 1$ ([BePaPa]). □

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Extension to other operators

Consider the second-order partial differential operator $\mathcal{L}$ defined by

$$\mathcal{L}f(z) = \sum_{i,j=1}^{n} a_{ij}(z) \frac{\partial^2 f}{\partial z_i \partial z_j}(z) + \sum_{i=1}^{n} a_i(z) \frac{\partial f}{\partial z_i}(z), \tag{7}$$

where the coefficients a_{ij} are smooth and homogeneous of degree $k \in [0, 1]$, i.e.

$$a_{ij}(\rho z) = \rho^k a_{ij}(z), \qquad 0 \le \rho \le 1, z \in \mathbb{U}, 1 \le i, j \le n, \tag{8}$$

and the coefficients a_i are also smooth and homogeneous of degree $k - 1$, i.e.

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If u and U are related by (6), it can be checked that

$$\mathcal{L}U(z) = \int_0^1 \rho^{1-k} \mathcal{L}u(\rho z)\, d\rho \quad \text{and} \quad \frac{\partial U}{\partial \nu}(z) = u(z), \qquad z \in \mathbb{U}.$$

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The previous observation leads us to the following more general result.

Theorem 2 ([BePaPa])

Assume $\phi : \partial\mathbb{U} \to \mathbb{R}$ is continuous. If u is the solution of the Dirichlet problem

$$\begin{cases} \mathcal{L}u = 0 \text{ in } D \\ u = \phi \text{ on } \partial D \end{cases} \tag{10}$$

where $\mathcal{L}$ is the operator given by (7) which satisfies (8) and (9), then

$$U(z) = \int_0^1 \frac{u(\rho z) - u(0)}{\rho} d\rho, \qquad z \in \mathbb{U}, \tag{11}$$

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From Neumann to the Dirichlet problem

A similar result can be given for the other direction, from the Neumann to the Dirichlet problem. As it is known, for a harmonic function the Laplacian and the partial derivatives commute. This observation allows to write the solution of the Dirichlet problem in terms of the solution of the Neumann problem, as follows.

Theorem 3 ([BePaPa])

Assume $\varphi : \partial\mathbb{U} \to \mathbb{R}$ is continuous and let U be the solution of the Neumann problem (2) with boundary condition $\phi = \varphi - \int_{\partial\mathbb{U}} \varphi(\xi)\,\sigma(d\xi)$. If we define

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An explicit representation of Dirichlet-to-Neumann operator

As an application of Theorem 1, we derive an explicit representation of the inverse of the Dirichlet-to-Neumann operator Λ_n in the case of the unit ball $\mathbb{U} \subset \mathbb{R}^n$ (a particular case of the Poincaré-Steklov operator), defined as follows.

For $\varphi \in C^\infty(\partial\mathbb{U})$, let u^φ be the solution to the Dirichlet problem (1) with boundary values φ.

The Dirichlet-to-Neumann operator Λ_n is the operator which maps the Dirichlet boundary values φ of the harmonic function u^φ in $\mathbb{U}$ to the corresponding Neumann boundary values $\frac{\partial u^\varphi}{\partial \nu}$ on $\partial\mathbb{U}$, i.e.

$$\Lambda(\varphi) = \left.\frac{\partial u^\varphi}{\partial \nu}\right|_{\partial\mathbb{U}}. \tag{14}$$

It can be checked that $\Lambda_n(\varphi_1) = \Lambda(\varphi_2) \Leftrightarrow \varphi_1 - \varphi_2 \equiv \text{const in } \mathbb{U}$, so if we consider

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Using the connection between the Dirichlet and Neumann problems given in Theorem 1, we can derive an explicit representation of the inverse of the operator Λ_n as follows.

Theorem 4 ([BePaPa])

Assume $\phi : \partial\mathbb{U} \to \mathbb{R}$ is continuous and satisfies $\int_{\partial\mathbb{U}} \phi(\xi)\,\sigma(d\xi) = 0$. We have

$$\Lambda_n^{-1}(\phi)(z) = \int_{\partial\mathbb{U}} \phi(\xi)\, k_n(z,\xi)\sigma_0(d\xi), \qquad z \in \partial\mathbb{U}, \tag{16}$$

where $k_n(z,\zeta) = \int_0^1 \frac{1}{\rho}\left(\frac{1-\rho^2}{|\rho z - \xi|^n} - 1\right) d\rho$, $z, \xi \in \partial\mathbb{U}$.

Explicitly, $k_2(z,\xi) = -2\ln|z-\xi|$, $k_3(z,\xi) = \frac{2}{|z-\xi|} - 2 + \ln 2 - \ln\left(\frac{|z-\xi|^2}{2} + |z-\xi|\right)$, *and for $n > 4$ the kernel $k_n(z,\xi)$ can be computed using the recurrence formulae*

$$k_n(z,\xi) = k_{n-2}(z,\xi) + \frac{2\left(1-|z-\xi|^{n-2}\right)}{(n-2)|z-\xi|^{n-2}} - \frac{1-|z-\xi|^{n-4}}{(n-4)|z-\xi|^{n-4}} + \left(1 - \frac{|z-\xi|^2}{2}\right) J_{n-2}(z,\xi), \tag{17}$$

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where $J_n(z,\xi) = \int_0^1 \frac{1}{|\rho z - \xi|^n} d\rho$.

Using the connection between the Dirichlet and Neumann problems given in Theorem 1, we can derive an explicit representation of the inverse of the operator Λ_n as follows.

Theorem 4 ([BePaPa])

Assume $\phi : \partial\mathbb{U} \to \mathbb{R}$ *is continuous and satisfies* $\int_{\partial\mathbb{U}} \phi(\xi)\,\sigma(d\xi) = 0$. *We have*

$$\Lambda_n^{-1}(\phi)(z) = \int_{\partial\mathbb{U}} \phi(\xi)\,k_n(z,\xi)\sigma_0(d\xi), \qquad z \in \partial\mathbb{U}, \tag{16}$$

where $k_n(z,\zeta) = \int_0^1 \frac{1}{\rho}\left(\frac{1-\rho^2}{|\rho z-\xi|^n} - 1\right)d\rho$, $z,\xi \in \partial\mathbb{U}$.

Explicitly, $k_2(z,\xi) = -2\ln|z-\xi|$, $k_3(z,\xi) = \frac{2}{|z-\xi|} - 2 + \ln 2 - \ln\left(\frac{|z-\xi|^2}{2} + |z-\xi|\right)$,

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Extension to general smooth bounded planar domains

Theorem 5 ([BePaPa])

Let $D \subset \mathbb{C}$ be a smooth bounded simply connected domain ($C^{1,\alpha}$ boundary with $0 < \alpha < 1$ will suffice), and for an arbitrarily fixed $w_0 \in D$ let $f : \mathbb{U} \to D$ be the conformal map of the unit disk $\mathbb{U}$ onto D with $f(0) = w_0$, $\arg f'(0) = 0$, and let $g = f^{-1} : D \to \mathbb{U}$ be its inverse.
Assume $\phi : \partial D \to \mathbb{R}$ is continuous and satisfies $\int_{\partial D} \phi(w)\, \sigma(dw) = 0$. If u is the solution of the Dirichlet problem (1) with boundary condition

$$\varphi(w) = \frac{1}{|g'(w)|} \phi(w), \qquad w \in \partial D, \tag{19}$$

then

$$U(w) = \int_0^1 \frac{u(f(\rho g(w)))}{\rho} d\rho, \qquad w \in D, \tag{20}$$

is the solution to the Neumann problem (2) with $U(w_0) = 0$.

Abstract Wiener space: preliminaries

Let (H, B) be an abstract Wiener space ([Gr], [Go]), and let $(W_t)_{t\geq 0}$ be the standard Wiener process (BM) with state space B.

For an open set $V \subset B$ we denote $\tau_x^V = \inf\{t \geq 0 : x + W_t \notin V\}$.

When $V = \mathbb{U}_r(0) = \{x \in B : \|x\| < r\}$ is a ball in B, we write $\tau_x^{(r)}$ for $\tau_x^{\mathbb{U}_r(0)}$, and τ_x if $r = 1$. When $x = 0$, we will omit this index.

For a Borel measurable function $f : \mathbb{S}_r(x) \to \mathbb{R}$, the average of f over $\mathbb{S}_r(x)$ ([Go]) is defined by

$$(A_r f)(x) = \int_{\mathbb{S}_r(0)} f(x+y)\pi_r(dy), \tag{21}$$

whenever the integral exists, where $\mathbb{S}_r(x) = \partial\mathbb{U}_r(x)$, and π_r is the *central hitting measure* defined by $\pi_r(E) = P(W_{\tau^{(r)}} \in E)$.

A function $f : V \subset B \to \mathbb{R}$ defined on an open subset V of the Wiener space (H, B) is called *harmonic* if it is locally bounded, Borel measurable, finely continuous, and there exists $\rho > 0$ such that $(A_r f)(x) = f(x)$, for every $0 < r < \rho$ for which $\overline{\mathbb{U}_r(x)} \subset V$.

The *generalized Laplacian* ([Gr]) of a Borel measurable function $f : V \subset B \to \mathbb{R}$ defined on (H, B) at the point $x \in V$ is defined by

$$\Delta f(x) = 2 \lim_{r \searrow 0} \frac{(A_r f)(x) - f(x)}{E\tau^{(r)}}, \tag{22}$$

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Abstract Wiener space: preliminaries

Remark 2

The strong Markov property of the infinite-dimensional Brownian motion shows that if V is strongly regular and if $f : \partial V \to \mathbb{R}$ is bounded and continuous, the stochastic solution of the Dirichlet problem for V with boundary values f given by

$$u(x) = E\left(f(x + W_{\tau_x^V}) 1_{\tau_x^V < \infty}\right), \qquad x \in V, \tag{23}$$

is a continuous, harmonic function in V, and has limiting boundary values f on ∂V.

Extension to the infinite-dimensional ball on an abstract Wiener space

The generalized Dirichlet and Neumann problems for the generalized Laplace operator Δ on a smooth open set $V \subset B$ is the problem of finding a continuous function $u : \overline{V} \to \mathbb{R}$ satisfying

$$\begin{cases} \Delta u = 0 \text{ in } V \\ u = \varphi \text{ on } \partial V \end{cases}, \tag{24}$$

respectively

$$\begin{cases} \Delta u = 0 \text{ in } V \\ \frac{\partial u}{\partial \nu} = \phi \text{ on } \partial V \end{cases}, \tag{25}$$

where $\nu(x)$ denotes the outward unit normal to the boundary of V at $x \in \partial V$.

The main result is the following.

Theorem 6 ([BePaPa])

Assume $\phi : \mathbb{S}_1(0) \to \mathbb{R}$ satisfies $\int_{\mathbb{S}_1(0)} \phi(z)\,\sigma(dz) = 0$ and is Lipschitz continuous. If u is the stochastic solution of the Dirichlet problem (24) for $V = \mathbb{U}_1(0)$ with boundary condition $\varphi = \phi$ on $\mathbb{S}_1(0)$, then

$$U(z) = \int_0^1 \frac{u(\rho z)}{\rho} d\rho, \qquad z \in \mathbb{U}_1(0), \tag{26}$$

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One of the difficulties in proving the above result is that the function U is well-defined (smoothness of u at the origin). To do this, we first proved the following.

Lemma 7 ([BePaPa])

Let $(W_t)_{t\geq 0}$ be an infinite-dimensional Brownian motion starting at $W_0 = x \in \mathbb{U}_1(0) \setminus \{0\}$ and let $\tau = \inf\{t \geq 0 : W_t \notin \mathbb{U}_1(0)\}$. Setting $\varepsilon = 1 - \|x\|$, we have

$$P\left(\|W_\tau - W_0\| \leq 5\sqrt[4]{\varepsilon}\right) \geq 1 - \sqrt{\varepsilon}. \tag{27}$$

As a consequence, we derived the following result.

Corollary 8 ([BePaPa])

The stochastic solution of the Dirichlet problem (24) for $V = \mathbb{U}_1(0)$, with Lipschitz continuous Dirichlet boundary condition $\varphi : \mathbb{S}_1(0) \to \mathbb{R}$ satisfying $\int_{\mathbb{S}_1(0)} \varphi(z)\,\sigma(dz) = 0$ is Hölder continuous of order $\frac{1}{4}$ in $\mathbb{U}_1(0)$.

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Extension to general boundary data: preliminaries

Example 9 (Zaremba's example)

The Dirichlet problem for the punctured disk $D = \mathbb{U} - \{0\}$ and the (continuous) boundary data $\varphi \equiv 0$ on $\partial\mathbb{U}$ and $\varphi(0) = 1$ has no (classical) solution.

If u were such a solution, then u must be bounded on D, and has a harmonic continuation to $\mathbb{U}$. Since u has limit 0 at each point of $\partial\mathbb{U}$, it follows $u \equiv 0$, contradicting $u(0) = 1$.

The Perron-Wiener-Brelot approach to the Dirichlet problem provides a harmonic function H_φ on D (for $\varphi : \partial D \to \mathbb{R}$ in a large class of functions), which coincides with the classical solution of the Dirichlet problem for D with boundary data φ, when this exists.

In the absence of a classical solution, H_φ still has boundary values φ, for most points of continuity of φ.

It can be shown that if φ is *resolutive*, then

$$H_\varphi(x) = \int_D \varphi d\mu_x, \tag{28}$$

where μ_x is the *harmonic measure* relative to D and x.

Probabilistically, μ_x is just the exit distribution from D of the BM starting at x, thus

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The Perron-Wiener-Brelot approach to the Dirichlet problem provides a harmonic function H_φ on D (for $\varphi : \partial D \to \mathbb{R}$ in a large class of functions), which coincides with the classical solution of the Dirichlet problem for D with boundary data φ, when this exists.
In the absence of a classical solution, H_φ still has boundary values φ, for most points of continuity of φ.
It can be shown that if φ is *resolutive*, then

$$H_\varphi(x) = \int_D \varphi d\mu_x, \tag{28}$$

where μ_x is the *harmonic measure* relative to D and x.
Probabilistically, μ_x is just the exit distribution from D of the BM starting at x, thus

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Extension to general boundary data: preliminaries

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Cornea's controlled convergence approach

An alternate approach to the Dirichlet problem is due to A. Cornea, using the notion of of *controlled convergence* ([Co1], [Co2]).

It can be shown that Cornea's approach and the Perron-Wiener-Brelot approach for the generalized solution of the Dirichlet problem are equivalent.

More precisely, it can be shown that for measurable boundary data satisfying a natural integrability condition, both methods indicate that the generalized solution of the Dirichlet problem is given by the stochastic solution $H_\varphi^{\mathbb{U}}$ defined by (30) (see [Co1], [Co2], [ArGa], and [BeCoRö]).

Definition 10 (*Controlled convergence* (A. Cornea, [Co1], [Co2]))

Let $D \subset \mathbb{R}^n$ be a bounded open set, $\partial D \subset \Delta \subset \overline{D}$, $f : \partial D \to \mathbb{R}$ and $h, k : D \to \overline{\mathbb{R}}$, $k \geq 0$. The function h converges to f controlled by k (we write $h \overset{k}{\to} f$) if the following conditions hold: For any set $A \subset D$ and any point $z_0 \in \overline{A} \cap \Delta$ we have

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Definition 11 ([Co1], [Co2])

A generalized solution of the Dirichlet problem (1) is a harmonic function $u : D \to \mathbb{R}$ which converges to φ, controlled by a continuous, non-negative harmonic function $k : D \to \mathbb{R}_+$.

In [Co1], the author showed that in the case of the unit ball $D = \mathbb{U} \subset \mathbb{R}^n$, every function $f \in L^1(\partial\mathbb{U}, \sigma_0)$ is resolutive for the Dirichlet problem.
Moreover, the generalized solution coincides in fact with the stochastic solution, that is

$$u(z) = H_\varphi^{\mathbb{U}}(z) = E^z \varphi(B_\tau), \tag{30}$$

where $(B_t)_{t\geq 0}$ is a n-dimensional BM starting at z and $\tau = \tau_z^{\partial\mathbb{U}} = \inf\{t \geq 0 : B_t \in \partial\mathbb{U}\}$ is the hitting time of the boundary of $\mathbb{U}$.
It is also known that the generalized solution of the Dirichlet problem is unique.

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Definition 12

Let $D \subset \mathbb{R}^d$ be a bounded open set, $\partial D \subset \Delta \subset \overline{D}$, $h, k : D \to \overline{\mathbb{R}}$, $k \geq 0$. We say that the function *h has a continuous extension to $\overline{D}$ controlled by k* if the following conditions hold:
for any set $A \subset D$ and any point $z_0 \in \overline{A} \cap \Delta$ we have

(i) If $\limsup_{A \ni z \to z_0} k(z) < +\infty$, we have $h(z_0) := \lim_{A \ni z \to z_0} h(z) \in \mathbb{R}$.

(ii) If $\lim_{A \ni z \to z_0} k(z) = +\infty$, then $\lim_{A \ni z \to z_0} \frac{h(z)}{1+k(z)} = 0$.

Remark. If h has a continuous extension to $\overline{D}$ controlled by k, then the function h can be extended by continuity on the set $\left\{z_0 \in \partial D : \limsup_{D \ni z \to z_0} k(z) < +\infty\right\}$. On the exceptional set $E = \left\{z_0 \in \partial D : \limsup_{D \ni z \to z_0} k(z) = +\infty\right\}$, the limit $\lim_{D \ni z \to z_0} h(z)$ may not exist, and the function h may fail to be continuous (this set of points is "controlled" by the function k).

Definition 13

A *generalized solution of the Neumann problem* (2) is a harmonic function $U : D \to \mathbb{R}$ which has a continuous extension to ∂D, controlled by a non-negative harmonic function $k : D \to \mathbb{R}_+$, and for any $z_0 \in \partial D$ for which $\limsup_{[0,z_0] \ni z \to z_0} k(z) < +\infty$ we have

$$\lim_{\varepsilon \nearrow 0} \frac{U(z_0 + \varepsilon \nu(z_0)) - U(z_0)}{\varepsilon} = \phi(z_0),$$

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Theorem 14

Assume $\phi : \partial\mathbb{U} \to \mathbb{R}$ *is integrable and satisfies* $\int_{\partial\mathbb{U}} \phi(z)\,\sigma_0(dz) = 0$*. If* u *is the generalized solution of the Dirichlet problem (1) with boundary condition* $\varphi = \phi$ *on* $\partial\mathbb{U}$*, then*

$$U(z) = \int_0^1 \frac{u(\rho z)}{\rho} d\rho, \qquad z \in \overline{\mathbb{U}}, \tag{31}$$

is a generalized solution to the Neumann problem (2) with $U(0) = 0$*.*

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