Convection-Diffusion Equations with Random Initial Conditions

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Laws of physics are descibed by simple differential equations.

M. Bąk

Sometimes it's hard to measure the initial conditions or other coefficients precisely, e.g.

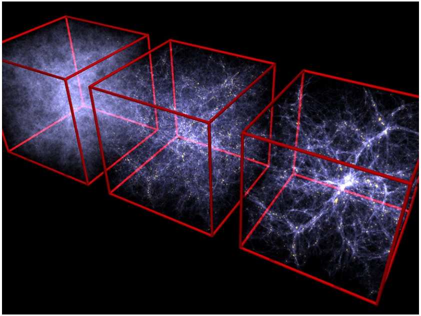
- Large Scale Structure of the Universe
- Ballistic deposition of chemical vapor for growing crystals (like diamonds)
- Saturation of rocks/soil with water/CO_2/hydrocarbons
- Polymers, traffic, queues, finance, etc.

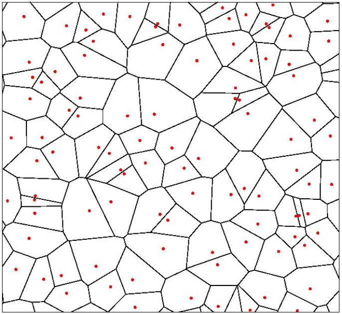
And many of the answers we seek are of "on average" type anyway.

Introduction II

The Universe as observed from any planet looks much the same.

Copernican Principle (by K. Rudnicki)





Homogenous/ isotropic/ motion invariant random fields

Fix a probability space (Ω, Σ, P).
Let $L^p(\Omega)$ denote the space of random variables s.t. $E\,X = 0$ and $\|X\|_p^p = E|X|^p < \infty$.
Let $u, v : \mathbb{R}^d \to L^p(\Omega)$. We say $u \stackrel{d}{=} v$ if

$$\forall n, x_1, \ldots, x_n \qquad \big(u(x_1), \ldots, u(x_n)\big) \stackrel{d}{=} \big(v(x_1), \ldots, v(x_n)\big) \qquad \text{as prob. distr. of r.v.}$$

For any isometry $\varphi \in \Phi$ on $\mathbb{R}$ define $\varphi(u)(x) = u(\varphi(x))$.

Space of "motion invariant" random fields:

$$\mathcal{MI}(L^p(\Omega)) = \{u \in C_b(\mathbb{R}, L^p(\Omega)) \ : \ \forall \varphi \in \Phi \ \varphi(u) \stackrel{d}{=} u\}.$$

Properties ($u \in \mathcal{MI}(L^2(\Omega))$):

- $u(x) = \int_{\mathbb{R}} e^{ix\xi} Z(d\xi)$, where Z is a random orthogonal measure.
- If Z is a ROM, then exists a measure σ s.t. $E\Big|\int f(\xi) Z(d\xi)\Big|^2 = \int |f(\xi)|^2 \sigma(d\xi)$.
- $B(|x-y|) = E\,u(x)u(y)$ depends only on $|x-y|$.
- B is the Fourier transform of σ.

Time-space random fields

Bielecki norm

$$u \in C(\mathbb{R}^+, C_b(\mathbb{R}, L^p(\Omega))) \quad |||u|||_K = \sup_{t\in[0,\infty)} e^{-tK} \sup_{x\in\mathbb{R}} \|u(t,x)\|_p.$$

Let $\varphi \in \Phi$ be an isometry on $\mathbb{R}$. We define

$$\varphi(u)(t,x) = u(t,\varphi(x)).$$

Jointly motion invariant random fields

$$\mathcal{JMI}(L^p(\Omega)) = \{u \in C(\mathbb{R}^+, \mathcal{MI}(L^p(\Omega))) \ : \ |||u|||_K < \infty \text{ and } \forall\varphi \in \Phi \ \varphi(u) \stackrel{d}{=} u\}.$$

Neither $\mathcal{MI}$ nor $\mathcal{JMI}$ are linear spaces! (Though they remain normed and complete.)

Let $u \in \mathcal{JMI}(L^p(\Omega))$. Then

- $u(t) + u(s) \in \mathcal{MI}(L^p(\Omega))$
- $\mathcal{L}u \in \mathcal{JMI}(L^p(\Omega))$ for a multiplier operator $\mathcal{L}$.

Let $u, v \in \mathcal{JMI}(L^p(\Omega))$ and suppose $(u,v) \stackrel{d}{=} (\varphi(u), \varphi(v))$.
We write $u \sim v$ (this relation is **not** transitive)

Linear equation

$$\begin{cases} \partial_t u = \mathcal{L} u & \text{in } [0,\infty) \times \mathbb{R}, \\ u(0) \overset{d}{=} u_0 & \text{on } \mathbb{R}. \end{cases}$$

Theorem

Suppose $\mathcal{L}$ is such that $\widehat{\left(e^{t\mathcal{L}} f\right)} = e^{tm(x)} \widehat{f}$, where $m : \mathbb{R} \to \mathbb{C}$ is continuous and $\operatorname{Re} m \le 0$.
Assume $u_0 \in \mathcal{MI}(L^2(\Omega))$.
The solution $u(t) = e^{t\mathcal{L}} u(0)$ is well defined, unique and jointly motion invariant.

- The semigroup $e^{t\mathcal{L}}$ does have a regularization effect in x, **but not** in ω.

Burgers equation (present state of the art)

$$\begin{cases} \partial_t u = \Delta u + \partial_x u^2 & \text{in } [0,\infty)\times\mathbb{R}, \\ u(0) \overset{d}{=} u_0 & \text{on } \mathbb{R}. \end{cases}$$

Theorem (Forsyth–Florin-Hopf-Cole)

The solution $u(t,x)$ to the Burgers equation may be represented as a transformation of a solution to the heat equation

$$\begin{cases} u(t,x) = -2\nabla \log q(t,x), \\ \partial_t q = \Delta q, \\ q(0) = \exp(-u_0/2). \end{cases}$$

- We can use this transformation to represent the solution to our problem in an explicit way, based on the result from the previous slide.

Fractional Burgers equation (part I)

$$\begin{cases} \partial_t u = -(-\Delta)^s u + \partial_x f(u) & \text{in } [0,\infty) \times \mathbb{R}, \\ u(0) \overset{d}{=} u_0 & \text{on } \mathbb{R}. \end{cases}$$

Define an operator

$$F(u) = P_t u(0) + \int_0^t \partial_x P_{t-\tau} f(u(s))\, d\tau$$

Definition

We say that u is a solution if $u(0) \overset{d}{=} u_0$ and $u = F(u)$.

Theorem

Let f be Lipschitz.
Assume $u_0 \in \mathcal{MI}(L^2(\Omega))$.
There exists a solution $u \in \mathcal{JMI}(L^2(\Omega))$.

Proof.

Define the sequence of Picard iterations $u_1 = P_t u_0$, $u_n = F(u_{n-1})$ (observe $u_n \sim u_m$).
Use Banach fixed point theorem (Lipschitz condition is crucial!). □

Energy estimates

Lemma (Rosenblatt '68)

Let u be a solution. Then

$$\partial_t E\, u(t)^2 = -2E\left((-\Delta)^{s/2} u(t)\right)^2$$

Proof.

$$\partial_t u^2 = 2uu_t = 2u(u_{xx} + f(u)_x) = 2(uu_x)_x - 2u_x^2 + 2uf(u)_x$$

$$uf(u)_x = uf'(u)u_x = h(u)_x, \quad \text{where } h(x) = \int_0^x yf'(y)\, dy.$$

Finally

$$\partial_t E\, u^2 = -2E\, u_x^2 + E(\ldots)_x = -2E\, u_x^2 + \partial_x E(\ldots) = -2E\, u_x^2.$$

□

- Same argument for all p-norms.

L^1-estimates

Lemma

Let $u \sim v$ be solutions and f be monotone. Then

$$\partial_t E\,|u(t) - v(t)| \leq 0$$

Proof.

$$\begin{aligned}\partial_t E\,|u - v| &= E\,(u_t - v_t)\,\mathrm{sgn}(u - v)\\ &= -E\,(-\Delta)^s(u - v)\,\mathrm{sgn}(u - v) + E\,\partial_x[f(u) - f(v)][\mathrm{sgn}(f(u) - f(v))] \leq 0\end{aligned}$$

□

Corollary (*)

$$E\,|u(t) - v(t)| \leq E\,|u_0 - v_0|$$

Approximate solutions

Recall

$$F(u) = P_t u(0) + \int_0^t \partial_x P_{t-\tau} f(u(s))\, d\tau$$

Notice that when $f(x) = x^2$, then $F : \mathcal{JMI}(L^2(\Omega)) \not\to \mathcal{JMI}(L^2(\Omega))$.

Let

$$u_0^n = u_0 \wedge n = \min(\max(u_0, -n), n)$$

be truncated initial conditions.

Notice that we have $L^\infty(\Omega)$ bound, so we are back in the Lipschitz case.

Consider the sequence of solutions u^n coming from the initial conditions u_0^n.

Corollary (* again)

$$E\,|u^n(t) - u^m(t)| \leq E\,|u_0^n - u_0^m| \to_{n,m} 0$$

Fractional Burgers equation (part II)

$$\begin{cases} \partial_t u = -(-\Delta)^s u + \partial_x f(u) & \text{in } [0,\infty) \times \mathbb{R}, \\ u(0) \overset{d}{=} u_0 & \text{on } \mathbb{R}. \end{cases}$$

Theorem

Let $f(x) = x^2$.
Assume $u_0 \in \mathcal{MI}(L^p(\Omega))$ for all $p < \infty$.
There exists a solution $u \in \bigcap \mathcal{JMI}(L^p(\Omega))$.

Proof.

Take the truncated solutions u^n.
Pass to the limit in $\mathcal{JMI}(L^1(\Omega))$, call it v.
Use Rosenblatt estimates in $L^p(\Omega)$ to establish uniform bounds.
Take $F(v)$ and show it is the pointwise limit of $F(u^n)$ in $L^1(\Omega)$. □

Thank you!