Kinetic equations and weighted inequalities

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Bedlewo Conference on Nonlocal Operators and PDE

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Kinetic equations (5%)
and integro differential inequalities (95%)

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Partly based on joint works with:

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2. Nader Masmoudi
3. Russell Schwab

Overview

Two problems

A pointwise inequality

*The classical ABP estimate (and a linear estimate)

Two problems

Problem #1: Sobolev for singular jump measures

An important problem is understanding the regularity of solutions to the equation given by an energy of the form

$$\mathcal{E}_\nu(u) := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(u(x)-u(y))^2\nu(x,dy)dx$$

Precisely: determine which ν's yield Hölder estimates, as well as Harnack inequality.

As we discussed earlier, the case $\nu \neq k(x,y)dy$ is very delicate.

Two problems

Problem #1: Sobolev for singular jump measures

Problem 1.
Characterize those ν's for which we have

$$\|u\|^2_{L^p(\mathbb{R}^N)} \leq C\mathcal{E}_\nu(u), \;\; p > 2.$$

To emphasize: particularly we want to consider ν's that may not have a density with respect to Lebesgue meausure.

Two problems

Problem #1: Sobolev for singular jump measures

The problem might become more tractable if one thinks about a harder one first: weighted Sobolev inequalities.

Problem 1'.
Characterize when does a weight v and measure ν are such that

$$\left(\int_{\mathbb{R}^N} v(x)|u(x)|^p \, dx\right)^{\frac{2}{p}} \leq C\mathcal{E}_\nu(u).$$

Of course, the previous problem is the case $v \equiv 1$.

Two problems

Weighted Sobolev inequalities

(Local) Weighted Sobolev inequalities $(v, w \geq 0)$

$$\left(\int v(x) |\phi(x)|^q \, dx \right)^{2/q} \leq C \int w(x) |\nabla \phi(x)|^2 \, dx$$

A theorem of Sawyer–Wheeden (1992) says: Suppose $2 < q$ and that v and w satisfy a certain doubling condition, then the above holds if and only if

$$\sup_B |B|^{\frac{1}{n}-1} \left(\int_B v(x) \, dx \right)^{\frac{1}{q}} \left(\int_B w(x)^{-1} \, dx \right)^{\frac{1}{2}} < \infty$$

This is known as the $\mathbf{A}_{p,q}$ **condition**.

Two problems

Weighted Sobolev inequalities

One idea is that by allowing $A(x)$ to be quite degenerate, equations such as

$$\int w(x)|\nabla u|^2 \, dx$$

may show a bigger variety of phenomena (estimates with different scalings). Then, second order theory gets a bit richer and closer (although still far) from the richness of

$$\int \int (u(x) - u(y))^2 \nu(x, dy) dx$$

It may be interesting to get a relation for ν and v similar to the $\mathbf{A}_{p,q}$ condition which determines the validity of a Sobolev ineq.

Two problems

Weighted Sobolev inequalities

Such inequalities are particularly useful when understanding linear divergence equations with a potential ("Schrödinger")

$$-\mathrm{div}(A(x)\nabla\phi) + v(x)\phi = 0,$$

where $A(x) \geq w(x)\mathrm{I}$, and $w(x)$ may become zero somewhere.

Two problems

Teaser for Problem 2.

Exercise: For a smooth, non-negative $f(v)$ we have

$$\int_{\mathbb{R}^d} f(v)^2 \, dv \leq 4 \int_{\mathbb{R}^d} a(v) |\nabla \sqrt{f(v)}|^2 \, dv$$

where $a = (-\Delta)^{-1} f$.

Two problems

Teaser for Problem 2.

Exercise: For a smooth, non-negative $f(v)$ we have

$$\int_{\mathbb{R}^d} f(v)^2 \, dv \leq 4 \int_{\mathbb{R}^d} a(v) |\nabla \sqrt{f(v)}|^2 \, dv$$

where $a = (-\Delta)^{-1} f$.

Proof: The integral on the right has the form

$$C(d) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |v-w|^{2-d} f(w) |\nabla \sqrt{f}(v)|^2 \, dw dv$$

Use the symmetry of the kernel + the arithmetic-geometric mean inequality, integrate by parts the resulting expression.

Two problems

Teaser for Problem 2.

Exercise: For $f(v)$ as before we have

$$\int_{\mathbb{R}^d} f(v)^2 \, dv \leq 4 \int_{\mathbb{R}^d} (A\nabla\sqrt{f(v)}, \nabla\sqrt{f(v)}) \, dv$$

where $A = A[f]$ is such that $\text{div}(\text{div} A) = -f$.

This inequality is equivalent to the monotonicity of the entropy in the homogeneous Landau equation.

(see Gressman–Krieger–Strain ’12)

Two problems

Problem #2: Best constants in a weighted inequality.

Problem 2.
Given a smooth, non-negative $f(v)$, prove

$$\int_{\mathbb{R}^d} f\phi^2 \, dv \leq C \int_{\mathbb{R}^d} a|\nabla\phi|^2 \, dv \ \ \forall \ \phi.$$

What can be said about the best constant $C = C(f)$?.

Note: This a very special kind of weighted Poincaré inequality!, since the weights are related by

$$-\Delta f = a.$$

An important and well studied instance is $f = |v|^{-m}$, where one gets Hardy inequalities.

Two problems

The Landau equation

The homogeneous Landau equation describes the evolution of a density $f = f(v,t)$ via a quadratic evolution equation

$$\partial_t f = Q(f,f)$$

Two problems

The Landau equation

The homogeneous Landau equation describes the evolution of a density $f = f(v,t)$ via a quadratic evolution equation

$$\partial_t f = Q(f,f)$$

Where $Q(f,f)$ is given by

$$\mathrm{div}\left(\int_{\mathbb{R}^3} \Phi(w-v)\Pi(w-v)\left(f(w)\nabla f(v) - f(v)\nabla f(w)\right)dw\right)$$

Here $\Phi(z) = c_\gamma |z|^{2+\gamma}$ $(\gamma \in [-3,0])$, $\Pi(z) = \mathrm{I} - \hat{z}\otimes\hat{z}$.

Two problems

The Landau equation

Moving the divergence inside the integral, the equation becomes

$$\partial_t f = \mathrm{div}(A\nabla f - f\nabla a) = \mathrm{tr}(AD^2 f) + hf$$

where

$$A = f * \left(c_\gamma |v|^{3+\gamma} \Pi(z)\right)$$
$$a = \mathrm{tr}(A) = (-\Delta)^{\frac{1+\gamma}{2}} f$$
$$h = -\Delta a = (-\Delta)^{\frac{3+\gamma}{2}} f$$

Two problems

The Landau equation

Consider the case $\gamma = -3$. Then, the equation takes the form

$$\begin{aligned} \partial_t f &= \mathrm{div}(A \nabla f - f \nabla a) \\ &= \mathrm{tr}(A D^2 f) + f^2. \end{aligned}$$

The quadratic term could potentially lead f to blow up.

Scaling wise, $\mathrm{tr}(AD^2 f) \approx a\, \Delta f$ –so, ideally, if f gets bad, the diffusion coefficient becomes strong.

Two problems

The Landau equation

Take $f(t=0) = f_{\text{in}}$ smooth initial data. It is well known that

$$\partial_t f = \Delta f + f^2 \Rightarrow \text{ finite time blow up.}$$

Meanwhile

$$\partial_t f = f\Delta f + f^2 \Rightarrow \text{ global smooth solutions.}$$

Krieger and Strain proposed the equation

$$\partial_t f = a\Delta f + \varepsilon f^2, \;\; \varepsilon \in [0,1],$$

and proved existence of a global smooth solution in the radial case, when $\varepsilon < 74/75$. The idea being that $\varepsilon = 1$ gives an isotropic analogue of the Landau equation.

Theorem (with Gualdani)

If $\varepsilon = 1$, the above equation has a global smooth solution if the initial data is symmetric and decreasing.

Two problems

The Landau equation

The proof also gives a conditional result for the Landau equation: f stays bounded up to $t = T$ as long as

$$\sup_r r^2 \frac{\int_{B_r} f(v,t)\,dv}{\int_{B_r} a_{f(\cdot,t)}(v)\,dv} < \frac{1}{96}, \;\; \forall \;\; t < T.$$

Recall that $a = (-\Delta)^{-1} f$.

Exercise. For *any* non-negative $f \in L^1$ and any r we have

$$r^2 \frac{\int_{B_r} f(v)\,dv}{\int_{B_r} a(v)\,dv} \leq 3.$$

Note: Such a condition *almost* implies a weighted inequality.

A pointwise inequality

Setup

Consider a measure metric space M, distance d, measure μ.

- There are constants c_0, c_1 such that

$$c_0 r^N \leq \mu(B_r(x)) \leq c_1 r^N \;\; \forall\, r < \operatorname{diam}(M).$$

- We are given a kernel $k(x,y)$ in M, and

$$k(x,y) \geq \lambda d(x,y)^{-N-\alpha},$$

  for some $\lambda, N > 0$ and $\alpha \in (0,2)$.

A pointwise inequality

Setup

The theorem involves an integral analogue of $|\nabla u(x)| \dots$

Given $u \in Lip(M)$, we define the function

$$|D|_k^2 u(x) := \int_M (u(x) - u(y))^2 \, k(x,y) d\mu(y)$$

Evidently, the integral of this function is equal to

$$\mathcal{E}_k(u) := \int_M \int_M (u(x) - u(y))^2 \, k(x,y) d\mu(x) d\mu(y).$$

A pointwise inequality

The main result

Theorem (forthcoming)

There is a constant C given by α, N, λ, and the c_i such that

$$|u(x)|^{2+\frac{\alpha q}{N}} \leq C\|u\|_{L^q(\mu)}^{\frac{\alpha q}{N}} |D|_k^2 u(x).$$

A pointwise inequality

The main result

Theorem (forthcoming)

There is a constant C given by α, N, λ, and the c_i such that

$$|u(x)|^{2+\frac{\alpha q}{N}} \leq C\|u\|_{L^q(\mu)}^{\frac{\alpha q}{N}} |D|_k^2 u(x).$$

Corollary

There is a constant C (same dependence as before) such that

$$\|u\|_{L^q(\mu)}^2 \leq C\mathcal{E}_\mu(u), \quad q := \frac{2N}{N-\alpha}.$$

A key lemma

The proof is based on an a very interesting lemma which was used in one of the key steps for the ABP-type estimate obtained with Schwab (2012).

Lemma

Suppose that $u : \mathbb{R}^d \to \mathbb{R}$ *satisfies*

$$-(-\Delta)^{\frac{\alpha}{2}} u(x_0) \leq f(x_0)$$

at a point x_0 *where* u *achieves its minimum.*

$$|u(x_0)| \leq C |f(x_0)|^{\frac{2-\alpha}{2}} |(-\Delta)^{-1+\frac{\alpha}{2}} u(x_0)|^{\frac{\alpha}{2}}.$$

The key lemma, adapted (1/3)

For u and x, define the "good set"

$$G_x := \{y \mid (u(y) - u(x))^2 \leq |D|_k^2 u(x) d(x,y)^\alpha\}$$

Fix $\rho > 0$. Given $k \in \mathbb{N}$, we will say it is *bad* if

$$\mu\left(R_{2^{-k}\rho} \setminus G_x\right) \geq \frac{1}{2}\mu(R_{2^{-k}\rho})$$

Lemma

There is a universal constant C such that

$$\#\{\text{ bad } k's\} \leq C.$$

Important: note the constant is independent of ρ.

The key lemma, adapted (2/3)

Proof. Fix $\rho > 0$, then

$$\begin{aligned} |D|_k^2 u(x) &= \int_M (u(x)-u(y))^2 k(x,y)\; d\mu(y) \\ &\geq \int_{M\setminus G_x} (u(x)-u(y))^2 k(x,y)\; d\mu(y) \\ &\geq \sum_{bad\;\; k's} \int_{R_{2^{-k}\rho}(x)\setminus G_x} (u(x)-u(y))^2 k(x,y)\; d\mu(y) \end{aligned}$$

To finish, we are going to show there is a universal constant ε_0 such that

$$\int_{R_{2^{-k}\rho}(x)\setminus G_x} (u(x)-u(y))^2 k(x,y)\; d\mu(y) \geq \varepsilon_0 |D|_k^2 u(x)$$

The key lemma, adapted (3/3)

It is here that we use most of our assumptions on $k(x, y)$ (writing $r := 2^{-k}\rho$)

$$\begin{aligned}
&\int_{R_r(x)\setminus G_x} (u(x) - u(y))^2 k(x, y)\, d\mu(y) \\
&\geq \lambda r^{-N-\alpha} \int_{R_r(x)\setminus G_x} |D|_k^2 u(x) d(x, y)^\alpha \, d\mu(y) \\
&\geq C^{-1} \lambda r^{-N} |D|_k^2 u(x) \mu(R_{2^{-k}\rho}(x) \setminus G_x).
\end{aligned}$$

The assumptions on $\mu(\cdot)$ imply

$$\mu(R_r(x) \setminus G_x) \geq C^{-1} r^N$$

Combining the inequalities, we are done.

Proof of the pointwise inequality

The lemma immediately implies there is a universal k_0 such that given u and x, then for some $k \leq k_0$ we have

$$\mu(R_{2^{-k}\rho}(x) \cap G_x) \geq \frac{1}{2}\mu(R_{2^{-k}\rho}(x)).$$

What choice of ρ yields useful information?

Let us choose ρ so u is not far from $u(x)$ in $B_\rho(x)$.

Proof of the pointwise inequality

Let us take

$$\rho = \left(\frac{|u(x)|^2}{2|D|_k^2 u(x)} \right)^{\frac{1}{\alpha}}$$

Then,

$$\begin{aligned} y \in B_{c\rho}(x) \cap G_x &\Rightarrow (u(x) - u(y))^2 \leq |D|_\mu^2 u(x) \rho^\alpha \\ &\Rightarrow |u(x) - u(y)| \leq \tfrac{1}{4} |u(x)| \\ &\Rightarrow |u(x)| \leq \tfrac{1}{4} |u(x)| + |u(y)|. \end{aligned}$$

Proof of the pointwise inequality

Conclusion: $|u(x)| \leq 2|u(y)|$ for y in good portion of $B_{c\rho}(x)$.

This information gives a relation between $\|u\|_{L^q}$ and $u(x)$.

$$\begin{aligned}\int_M |u(y)|^q \, d\mu(y) &\geq \int_{B_{c\rho}(x)} |u(y)|^q \, d\mu(y) \\ &\geq C^{-1}\mu(B_{c\rho})|u(x)|^q.\end{aligned}$$

Using again the assumption on μ,

$$\|u\|_q^q \geq C^{-1}\rho^N |u(x)|^q = C^{-1}\left(\frac{|u(x)|^2}{2|D|_k^2 u(x)}\right)^{\frac{N}{\alpha}} |u(x)|^q$$

Proof of the pointwise inequality

Rearranging, we get the desired inequality:

$$|u(x)|^{2+\frac{\alpha q}{N}} \leq C\|u\|_q^{\frac{\alpha q}{N}} |D|_k^2 u(x).$$

Let us see this inequality for $M = \mathbb{R}^N$ and $(-\Delta)^{\frac{\alpha}{2}}$,

$$|u(x)|^{2+\frac{\alpha q}{N}} \leq C\|u\|_{L^q(\mathbb{R}^N)}^{\frac{\alpha q}{N}} \int_{\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+\alpha}} dy$$

Proof of the Sobolev inequality

If $q = \frac{2N}{N-\alpha}$, the exponents simplify to

$$|u(x)|^q = |u(x)|^{\frac{2N}{N-\alpha}} \leq C\|u\|_q^{\frac{\alpha q}{N}} |D|_k^2 u(x).$$

Integrating,

$$\int_M |f(x)|^q \, d\mu(x) \leq C\|f\|_q^{\frac{q\alpha}{N}} \int_M D_k f(x) \, d\mu(x)$$
$$\Rightarrow \|f\|_q^{q\left(1-\frac{\alpha}{N}\right)} \leq C\mathcal{E}_k(u)$$

Since $q(1 - \frac{\alpha}{N}) = 2$, we obtain the inequality.

The ABP estimate

Let us now go to the non-divergence setting.

Consider the operator

$$Lu(x) = \int_{\mathbb{R}^d \setminus \{0\}} (u(x+y) - u(x))k(x,y)dy, \quad k(x,\cdot) \text{ even.}$$

We also consider the Dirichlet problem

$$Lu(x) = f(x) \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^d \setminus \Omega.$$

The ABP estimate

Dirichlet problem

$$Lu(x) = f(x) \text{ in } \Omega, \;\; u = 0 \text{ in } \mathbb{R}^d \setminus \Omega.$$

Problem: Find out for which k and p we have

$$\|u\|_{L^\infty(\Omega)} \leq C\|f\|_{L^p(\Omega)}$$

where C is independent of u, and in fact $C = C(k, \Omega, p)$.

(the dependence in K being indifferent to its regularity)

The ABP estimate

Example: $L = -(-\Delta)^{\frac{\alpha}{2}}$, $\Omega = B_1$.

From the formula for the fundamental solution, one gets

$$\|u\|_{L^\infty(\Omega)} \leq C\|f\|_{L^p(\Omega)}, \quad p > \frac{2d}{\alpha}.$$

For some $C = C(d, p, \alpha)$.

The ABP estimate

More concretely, we ask: for L of the form

$$k(x,y) = \frac{a(x,y)}{|y|^{d+\alpha}} \quad \text{with} \quad \lambda \le a \le \Lambda.$$

Can one prove an estimate

$$\|u\|_{L^\infty(\Omega)} \le C\|f\|_{L^p(\Omega)}$$

for some $p < \infty$ and C depending only on α, d, λ, and Λ?.

The classical ABP estimate

Consider the second order differential operator

$$L(u,x) = \mathrm{tr}(A(x)D^2u(x))$$

where A is smooth and $\lambda \mathrm{I} \le A \le \Lambda \mathrm{I}$, and the problem

$$\begin{aligned} Lu &= f \text{ in } \Omega \\ u &= 0 \text{ in } \partial\Omega \end{aligned}$$

Theorem (Aleksandrov–Bakelman–Pucci)

There is a constant $C = C(\lambda, d, |\Omega|)$ such that

$$\|u_-\|_{L^\infty(\Omega)} \le C\|f\|_{L^d(K_u)}$$

Where $K_u = \{x \in \Omega \mid u = \Gamma_u\}$.

ABP and second order equations

A very brief list of subsequent results using this estimate

1. Holder estimates for equations in non-divergence form
2. $C^{2,\alpha}$ regularity for convex fully nonlinear equations
3. Obstacle problem
4. Stochastic homogenization
5. $W^{2,p}$ theory for fully nonlinear equations
6. Integrability for the L-Green function
7. Strong maximum principles in small domains

ABP and integro-differential equations

L^∞ version is sufficient – L^p analogue is required

The available estimates for $\alpha < 2$ do not go as far.

1. Holder estimates for equations in non-divergence form
2. $C^{\alpha+\beta}$ regularity for convex fully nonlinear equations
3. Obstacle problem
4. Stochastic homogenization
5. $W^{\alpha,p}$ theory
6. Integrability for the L-Green function
7. Strong maximum principles

An ABP-type estimate for integro-diff. equations

Consider a very special type of kernel

$$k(x,y) = \frac{a(x,y)}{|y|^{d+\alpha}} \quad \text{where}$$
$$a(x,y) = (2-\alpha)(A(x)\hat{y},\hat{y}).$$

Where $A(x) \geq 0$ and $\operatorname{tr}(A(x)) \geq \lambda$ for all x.

Theorem (with Schwab, 2012)

There is a constant $C = C(\alpha, \lambda, d, |\Omega|)$ such that

$$\|u_-\|_{L^\infty(\Omega)} \leq C\|f\|_{L^\infty(K_{\alpha,u})}^{\frac{2-\alpha}{\alpha}} \|f\|_{L^d(K_{\alpha,u})}^{\frac{\alpha}{2}}.$$

Where $K_{\alpha,u} = \{x \in \Omega \mid u = \Gamma_{\alpha,u}\}$.

An improvement: removing the L^∞ dependence

At least for the linear Dirichlet problem, an interpolation argument leads to a bound in terms of just L^p.

Theorem (with N. Masmoudi)

For any $p > \frac{2d}{\alpha}$ *there is a* $C = C(\lambda, d, |\Omega|, p)$ *such that*

$$\|u_-\|_{L^\infty(\Omega)} \leq C\|f\|_{L^p(\Omega)}.$$

$\exists\ p < \frac{2d}{\alpha}$ *with* $p = p(\lambda, \Lambda, d)$ *and* $C = C(\lambda, \Lambda, d)$ *such that*

$$\|u_-\|_{L^\infty(\Omega)} \leq C\|f\|_{L^p(\Omega)}.$$

Proof in an ideal, simple world

Let us explain the proof of the more recent result, taking the original one with Schwab as a starting point.

Idea: If $\|f\|_{L^\infty} \leq 1$, the modified ABP+Hölder yields

$$\|u\|_\infty \leq C\|f\|_{L^d(\Omega)}^{\frac{\alpha}{2}} \Rightarrow \|u\|_\infty \leq C'\|f\|_{L^{\frac{2d}{\alpha}}(\Omega)}$$

To handle $\|f\|_{L^\infty}$ *in general*, express u as a sum, and interpolate with a L^p-norm of f with $p > 2d/\alpha$.

Proof in an ideal, simple world

Fix $f \geq 0$. The *layer decomposition* of f says that

$$f(x) = \int_0^\infty \chi_{E_r}(x)\, dr, \quad E_r := \{f > r\}.$$

Then, define u_r $(r > 0)$ as the solution* to the problem

$$Lu_r(x) = \chi_{E_r} \text{ in } \Omega, \;\; u_r = 0 \text{ in } \mathbb{R}^d \setminus \Omega.$$

The linearity of the equation, and the existence+uniqueness of the problem yields

$$u(x) = \int_0^\infty u_r(x)\, dr.$$

Proof in an ideal, simple world

Fix $R > 0$ and break the previous integral in two

$$\|u\|_\infty \leq \int_0^R \|u_r\|_\infty \, dr + \int_R^\infty \|u_r\|_\infty \, dr$$

Now, applying the modified ABP estimate

$$\|u_r\|_\infty \leq C \|\chi_{E_r}\|_{L^\infty(\Omega)}^{\frac{2-\alpha}{2}} \|\chi_{E_r}\|_{L^d(\Omega)}^{\frac{\alpha}{2}}.$$

Which can be used in two extreme ways, either

$$\|u_r\|_{L^\infty(\Omega)} \leq C, \quad \|u_r\|_{L^\infty(\Omega)} \leq C|E_r|^{\frac{\alpha}{2d}}$$

Proof in an ideal, simple world

We conclude that

$$\|u\|_\infty \leq CR + R^{\frac{\alpha p}{2d}} \|u\|_{L^p(\Omega)}^{1-\frac{\alpha p}{2d}}$$

Minimizing the right hand side in R we get the estimate.

Thank you!