Nonlocal space-time equations and the master equation

Pablo Raúl Stinga

Iowa State University

3rd Conference on Nonlocal Operators and Partial Differential Equations
Bedlewo, Poland

June 27th, 2016

Fractional nonlocal parabolic equations

Parabolic equation (almost as in Krylov's books)

$$Lu \equiv (\partial_t - a^{ij}(t)\partial_{ij})u = f(t,x) \qquad (t,x) \in \mathbb{R}^{n+1}$$

Coefficients depend only on t,

$$a^{ij}(t) = a^{ji}(t) \in L^\infty(\mathbb{R}) \qquad \lambda^{-1}I \leq (a^{ij}(t)) \leq \lambda I$$

for some $\lambda > 0$.

Fractional nonlocal parabolic equations

Parabolic equation (almost as in Krylov's books)

$$Lu \equiv (\partial_t - a^{ij}(t)\partial_{ij})u = f(t,x) \qquad (t,x) \in \mathbb{R}^{n+1}$$

Coefficients depend only on t,

$$a^{ij}(t) = a^{ji}(t) \in L^\infty(\mathbb{R}) \qquad \lambda^{-1} I \leq (a^{ij}(t)) \leq \lambda I$$

for some $\lambda > 0$.

For $0 < s < 1$ we look at the fractional equation

$$L^s u \equiv (\partial_t - a^{ij}(t)\partial_{ij})^s u = f(t,x)$$

Fractional nonlocal parabolic equations

Parabolic equation (almost as in Krylov's books)

$$Lu \equiv (\partial_t - a^{ij}(t)\partial_{ij})u = f(t,x) \qquad (t,x) \in \mathbb{R}^{n+1}$$

Coefficients depend only on t,

$$a^{ij}(t) = a^{ji}(t) \in L^\infty(\mathbb{R}) \qquad \lambda^{-1}I \leq (a^{ij}(t)) \leq \lambda I$$

for some $\lambda > 0$.

For $0 < s < 1$ we look at the fractional equation

$$L^s u \equiv (\partial_t - a^{ij}(t)\partial_{ij})^s u = f(t,x)$$

- *Fractional nonlocal parabolic equation*
- Joint work with J. L. Torrea (Univ. Autónoma de Madrid)
- Parabolic Signorini problem, master equation (CTRW)...

The language of semigroups (with J. L. Torrea, 2010)

If $\{e^{-\tau L}\}_{\tau\geq 0}$ is the **semigroup** generated by L then

$$L^s u = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{-\tau L} u - u\right) \frac{d\tau}{\tau^{1+s}}$$

- Gamma function: $$\lambda^s = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{-\tau \lambda} - 1\right) \frac{d\tau}{\tau^{1+s}}$$

The language of semigroups (with J. L. Torrea, 2010)

If $\{e^{-\tau L}\}_{\tau\geq 0}$ is the **semigroup** generated by L then

$$L^s u = \frac{1}{\Gamma(-s)}\int_0^\infty \left(e^{-\tau L}u - u\right)\frac{d\tau}{\tau^{1+s}}$$

- Gamma function: $$\lambda^s = \frac{1}{\Gamma(-s)}\int_0^\infty \left(e^{-\tau\lambda} - 1\right)\frac{d\tau}{\tau^{1+s}}$$

Example: the fractional Laplacian.

$$\begin{aligned}(-\Delta)^s u(x) &= \frac{1}{\Gamma(-s)}\int_0^\infty \left(e^{\tau\Delta}u(x) - u(x)\right)\frac{d\tau}{\tau^{1+s}} \\ &= c_s\int_0^\infty\int_{\mathbb{R}^n}\frac{e^{-|x-z|^2/(4\tau)}}{(4\pi\tau)^{n/2}}\big(u(z)-u(x)\big)\,dz\,\frac{d\tau}{\tau^{1+s}} \;\leftarrow\; (e^{\tau\Delta}1 = 1)\end{aligned}$$

The language of semigroups (with J. L. Torrea, 2010)

If $\{e^{-\tau L}\}_{\tau\geq 0}$ is the **semigroup** generated by L then

$$L^s u = \frac{1}{\Gamma(-s)}\int_0^\infty \left(e^{-\tau L}u - u\right)\frac{d\tau}{\tau^{1+s}}$$

- Gamma function: $\lambda^s = \frac{1}{\Gamma(-s)}\int_0^\infty \left(e^{-\tau\lambda} - 1\right)\frac{d\tau}{\tau^{1+s}}$

Example: the fractional Laplacian.

$$\begin{aligned}(-\Delta)^s u(x) &= \frac{1}{\Gamma(-s)}\int_0^\infty \left(e^{\tau\Delta}u(x) - u(x)\right)\frac{d\tau}{\tau^{1+s}} \\ &= c_s\int_0^\infty \int_{\mathbb{R}^n} \frac{e^{-|x-z|^2/(4\tau)}}{(4\pi\tau)^{n/2}}\left(u(z)-u(x)\right)dz\,\frac{d\tau}{\tau^{1+s}} \quad \leftarrow \quad (e^{\tau\Delta}1 = 1) \\ &= c_{n,s}\,\mathrm{P.V.}\int_{\mathbb{R}^n}\left(u(z)-u(x)\right)\left[\int_0^\infty \frac{e^{-|x-z|^2/(4\tau)}}{\tau^{n/2+s}}\frac{d\tau}{\tau}\right]dz\end{aligned}$$

The language of semigroups (with J. L. Torrea, 2010)

If $\{e^{-\tau L}\}_{\tau \geq 0}$ is the **semigroup** generated by L then

$$L^s u = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{-\tau L} u - u\right) \frac{d\tau}{\tau^{1+s}}$$

- Gamma function: $\lambda^s = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{-\tau\lambda} - 1\right) \frac{d\tau}{\tau^{1+s}}$

Example: the fractional Laplacian.

$$\begin{aligned}
(-\Delta)^s u(x) &= \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{\tau\Delta} u(x) - u(x)\right) \frac{d\tau}{\tau^{1+s}} \\
&= c_s \int_0^\infty \int_{\mathbb{R}^n} \frac{e^{-|x-z|^2/(4\tau)}}{(4\pi\tau)^{n/2}} \left(u(z) - u(x)\right) dz \, \frac{d\tau}{\tau^{1+s}} \quad \leftarrow \quad (e^{\tau\Delta} 1 = 1) \\
&= c_{n,s} \, \mathrm{P.V.} \int_{\mathbb{R}^n} \left(u(z) - u(x)\right) \left[\int_0^\infty \frac{e^{-|x-z|^2/(4\tau)}}{\tau^{n/2+s}} \, \frac{d\tau}{\tau} \right] dz \\
&= c_{n,s} \, \mathrm{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(z)}{|x-z|^{n+2s}} \, dz
\end{aligned}$$

The language of semigroups (with J. L. Torrea, 2010)

If $\{e^{-\tau L}\}_{\tau\geq 0}$ is the **semigroup** generated by L then

$$L^s u = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{-\tau L}u - u\right) \frac{d\tau}{\tau^{1+s}}$$

- Gamma function: $\lambda^s = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{-\tau\lambda} - 1\right) \frac{d\tau}{\tau^{1+s}}$

Example: the fractional Laplacian.

$$\begin{aligned}
(-\Delta)^s u(x) &= \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{\tau\Delta}u(x) - u(x)\right) \frac{d\tau}{\tau^{1+s}} \\
&= c_s \int_0^\infty \int_{\mathbb{R}^n} \frac{e^{-|x-z|^2/(4\tau)}}{(4\pi\tau)^{n/2}} \left(u(z) - u(x)\right) dz \, \frac{d\tau}{\tau^{1+s}} \quad \leftarrow \quad (e^{\tau\Delta}1 = 1) \\
&= c_{n,s} \, \mathrm{P.V.} \int_{\mathbb{R}^n} \left(u(z) - u(x)\right) \left[\int_0^\infty \frac{e^{-|x-z|^2/(4\tau)}}{\tau^{n/2+s}} \frac{d\tau}{\tau} \right] dz \\
&= c_{n,s} \, \mathrm{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(z)}{|x-z|^{n+2s}} \, dz
\end{aligned}$$

- Explicity constant $c_{n,s} > 0$, no Fourier transform used but **semigroup kernel**.

Fractional nonlocal parabolic equations

$$L^s u \equiv (\partial_t - a^{ij}(t)\partial_{ij})^s u$$

Fractional nonlocal parabolic equations

$$L^s u \equiv (\partial_t - a^{ij}(t)\partial_{ij})^s u$$

We need the semigroup

$$v(\tau, t, x) \equiv e^{-\tau L} u(t, x)$$

so that

$$\begin{cases} \partial_\tau v = -(\partial_t - a^{ij}(t)\partial_{ij}) v, & \text{for } \tau > 0, \\ v\big|_{\tau=0} = u(t, x), & \text{for } (t, x) \in \mathbb{R}^{n+1}. \end{cases}$$

Fractional nonlocal parabolic equations

$$L^s u \equiv (\partial_t - a^{ij}(t)\partial_{ij})^s u$$

We need the semigroup

$$v(\tau, t, x) \equiv e^{-\tau L} u(t, x)$$

so that

$$\begin{cases} \partial_\tau v = -(\partial_t - a^{ij}(t)\partial_{ij})v, & \text{for } \tau > 0, \\ v\big|_{\tau=0} = u(t,x), & \text{for } (t,x) \in \mathbb{R}^{n+1}. \end{cases}$$

- Notice that

$$\partial_t \quad \text{and} \quad a^{ij}(t)\partial_{ij}$$

do not commute, so

$$e^{-\tau(\partial_t - a^{ij}(t)\partial_{ij})} \neq e^{-\tau\partial_t} \circ e^{-\tau a^{ij}(t)\partial_{ij}}$$

The semigroup

With

$$A(r,t) := \int_r^t a(\rho)\,d\rho \sim (t-r)Id$$

and

$$\sigma_{rt} := A(r,t)^{1/2} \sim (t-r)^{1/2} Id$$

The semigroup

With

$$A(r,t) := \int_r^t a(\rho)\,d\rho \sim (t-r)Id$$

and

$$\sigma_{rt} := A(r,t)^{1/2} \sim (t-r)^{1/2} Id$$

we can write

$$\begin{aligned} e^{-\tau L}u(t,x) &= e^{A^{ij}(t-\tau,t)\partial_{ij}}u(t-\tau,x) \\ &= \int_{\mathbb{R}^n} p(t,t-\tau,y)u(t-\tau,x+y)\,dy \qquad \tau>0, \end{aligned}$$

where

$$p(t,r,y) = \chi_{r<t}\frac{e^{-|\sigma_{rt}^{-1}y|^2/4}}{(4\pi)^{n/2}(\det\sigma_{rt})} \sim \chi_{r<t}\frac{e^{-|y|^2/(4(t-r))}}{(4\pi(t-r))^{n/2}}$$

The semigroup

With

$$A(r,t) := \int_r^t a(\rho)\,d\rho \sim (t-r)Id$$

and

$$\sigma_{rt} := A(r,t)^{1/2} \sim (t-r)^{1/2} Id$$

we can write

$$\begin{aligned} e^{-\tau L}u(t,x) &= e^{A^{ij}(t-\tau,t)\partial_{ij}}u(t-\tau,x) \\ &= \int_{\mathbb{R}^n} p(t,t-\tau,y)u(t-\tau,x+y)\,dy \qquad \tau > 0, \end{aligned}$$

where

$$p(t,r,y) = \chi_{r<t}\frac{e^{-|\sigma_{rt}^{-1}y|^2/4}}{(4\pi)^{n/2}(\det\sigma_{rt})} \sim \chi_{r<t}\frac{e^{-|y|^2/(4(t-r))}}{(4\pi(t-r))^{n/2}}$$

Moreover,

$$e^{-\tau L}1(t,x) = 1, \qquad \tau > 0,\ (t,x) \in \mathbb{R}^{n+1}$$

The semigroup

With

$$A(r,t) := \int_r^t a(\rho)\,d\rho \sim (t-r)Id$$

and

$$\sigma_{rt} := A(r,t)^{1/2} \sim (t-r)^{1/2}Id$$

we can write

$$\begin{aligned} e^{-\tau L}u(t,x) &= e^{A^{ij}(t-\tau,t)\partial_{ij}}u(t-\tau,x) \\ &= \int_{\mathbb{R}^n} p(t,t-\tau,y)u(t-\tau,x+y)\,dy \qquad \tau>0, \end{aligned}$$

where

$$p(t,r,y) = \chi_{r<t}\frac{e^{-|\sigma_{rt}^{-1}y|^2/4}}{(4\pi)^{n/2}(\det\sigma_{rt})} \sim \chi_{r<t}\frac{e^{-|y|^2/(4(t-r))}}{(4\pi(t-r))^{n/2}}$$

Moreover,

$$e^{-\tau L}1(t,x) = 1, \qquad \tau>0,\ (t,x)\in\mathbb{R}^{n+1}$$

- We can apply the *semigroup language* to $L = \partial_t - a^{ij}(t)\partial_{ij}$

Fractional nonlocal parabolic equations

$$\begin{aligned}(\partial_t - a^{ij}(t)\partial_{ij})^s u(t,x) &= \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{-\tau L}u(t,x) - u(t,x)\right) \frac{d\tau}{\tau^{1+s}} \\ &= \int_0^\infty \int_{\mathbb{R}^n} \left(u(t,x) - u(t-\tau,x+y)\right) \frac{p(t,t-\tau,y)}{|\Gamma(-s)|\,\tau^{1+s}}\,dy\,d\tau\end{aligned}$$

Fractional nonlocal parabolic equations

$$(\partial_t - a^{ij}(t)\partial_{ij})^s u(t,x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{-\tau L}u(t,x) - u(t,x)\right) \frac{d\tau}{\tau^{1+s}}$$
$$= \int_0^\infty \int_{\mathbb{R}^n} \left(u(t,x) - u(t-\tau, x+y)\right) \frac{p(t, t-\tau, y)}{|\Gamma(-s)|\ \tau^{1+s}}\, dy\, d\tau$$

Particular case: the fractional heat operator.

$$(\partial_t - \Delta)^s u(t,x) = \int_0^\infty \int_{\mathbb{R}^n} \left(u(t,x) - u(t-\tau, z)\right) K(t,x,\tau,z)\, dz\, d\tau$$

Fractional nonlocal parabolic equations

$$(\partial_t - a^{ij}(t)\partial_{ij})^s u(t,x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{-\tau L}u(t,x) - u(t,x)\right) \frac{d\tau}{\tau^{1+s}}$$

$$= \int_0^\infty \int_{\mathbb{R}^n} \left(u(t,x) - u(t-\tau, x+y)\right) \frac{p(t, t-\tau, y)}{|\Gamma(-s)|\, \tau^{1+s}}\, dy\, d\tau$$

Particular case: the fractional heat operator.

$$(\partial_t - \Delta)^s u(t,x) = \int_0^\infty \int_{\mathbb{R}^n} \left(u(t,x) - u(t-\tau, z)\right) K(t,x,\tau,z)\, dz\, d\tau$$

where $K(t,x,\tau,z) = K(\tau, x-z) = c_{n,s} \dfrac{e^{-|x-z|^2/(4\tau)}}{\tau^{n/2+1+s}}$ satisfies

- $K(t,x,\tau,z) \geq \dfrac{\lambda}{|x-z|^{n+2+2s}}$, when $\tau \sim |x-z|^2$
- $K(t,x,\tau,z) \leq \dfrac{\Lambda}{|x-z|^{n+2+2s} + \tau^{n/2+1+s}}$

Fractional nonlocal parabolic equations

$$(\partial_t - a^{ij}(t)\partial_{ij})^s u(t,x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{-\tau L}u(t,x) - u(t,x)\right) \frac{d\tau}{\tau^{1+s}}$$
$$= \int_0^\infty \int_{\mathbb{R}^n} \left(u(t,x) - u(t-\tau, x+y)\right) \frac{p(t, t-\tau, y)}{|\Gamma(-s)|\ \tau^{1+s}}\, dy\, d\tau$$

Particular case: the fractional heat operator.

$$(\partial_t - \Delta)^s u(t,x) = \int_0^\infty \int_{\mathbb{R}^n} \big(u(t,x) - u(t-\tau, z)\big) K(t,x,\tau,z)\, dz\, d\tau$$

where $K(t,x,\tau,z) = K(\tau, x-z) = c_{n,s} \dfrac{e^{-|x-z|^2/(4\tau)}}{\tau^{n/2+1+s}}$ satisfies

- $K(t,x,\tau,z) \geq \dfrac{\lambda}{|x-z|^{n+2+2s}}$, when $\tau \sim |x-z|^2$
- $K(t,x,\tau,z) \leq \dfrac{\Lambda}{|x-z|^{n+2+2s} + \tau^{n/2+1+s}}$

▶ Master equation, as in Caffarelli–Silvestre (2014)

Extension problem (with J. L. Torrea, 2010)

If U solves

$$\begin{cases} -LU + \frac{1-2s}{y} U_y + U_{yy} = 0, & \text{for } y > 0, \\ U\big|_{y=0^+} = u, \end{cases}$$

then

$$-y^{1-2s}\partial_y U\big|_{y=0^+} = c_s L^s u$$

Extension problem (with J. L. Torrea, 2010)

If U solves

$$\begin{cases} -LU + \frac{1-2s}{y} U_y + U_{yy} = 0, & \text{for } y > 0, \\ U\big|_{y=0^+} = u, \end{cases}$$

then

$$-y^{1-2s} \partial_y U\big|_{y=0^+} = c_s L^s u$$

Explicit formulas.

$$U = \frac{y^{2s}}{4^s \Gamma(s)} \int_0^\infty e^{-y^2/(4\tau)} e^{-\tau L} u \, \frac{d\tau}{\tau^{1+s}} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-y^2/(4\tau)} e^{-\tau L} (L^s u) \, \frac{d\tau}{\tau^{1-s}}$$

Extension problem (with J. L. Torrea, 2010)

If U solves

$$\begin{cases} -LU + \frac{1-2s}{y} U_y + U_{yy} = 0, & \text{for } y > 0, \\ U\big|_{y=0^+} = u, \end{cases}$$

then

$$-y^{1-2s} \partial_y U\big|_{y=0^+} = c_s L^s u$$

Explicit formulas.

$$U = \frac{y^{2s}}{4^s \Gamma(s)} \int_0^\infty e^{-y^2/(4\tau)} e^{-\tau L} u \, \frac{d\tau}{\tau^{1+s}} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-y^2/(4\tau)} e^{-\tau L} (L^s u) \, \frac{d\tau}{\tau^{1-s}}$$

If $L^s u = f$, clearly U solves

$$\begin{cases} -LU + \frac{1-2s}{y} U_y + U_{yy} = 0, & \text{for } y > 0, \\ -y^{1-2s} \partial_y U\big|_{y=0^+} = f, \end{cases}$$

and then

$$U\big|_{y=0^+} = c_s^{-1} L^{-s} f$$

Extension problem (with J. L. Torrea, 2010)

If U solves

$$\begin{cases} -LU + \frac{1-2s}{y} U_y + U_{yy} = 0, & \text{for } y > 0, \\ U\big|_{y=0^+} = u, \end{cases}$$

then

$$-y^{1-2s}\partial_y U\big|_{y=0^+} = c_s L^s u$$

Explicit formulas.

$$U = \frac{y^{2s}}{4^s\Gamma(s)} \int_0^\infty e^{-y^2/(4\tau)} e^{-\tau L} u \, \frac{d\tau}{\tau^{1+s}} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-y^2/(4\tau)} e^{-\tau L}(L^s u) \, \frac{d\tau}{\tau^{1-s}}$$

If $L^s u = f$, clearly U solves

$$\begin{cases} -LU + \frac{1-2s}{y} U_y + U_{yy} = 0, & \text{for } y > 0, \\ -y^{1-2s}\partial_y U\big|_{y=0^+} = f, \end{cases}$$

and then

$$U\big|_{y=0^+} = c_s^{-1} L^{-s} f$$

- L: parabolic $\partial_t - a^{ij}(t)\partial_{ij}$ (with J. L. Torrea, 2016)

Extension problem (with J. L. Torrea, 2010)

If U solves

$$\begin{cases} -LU + \frac{1-2s}{y} U_y + U_{yy} = 0, & \text{for } y > 0, \\ U\big|_{y=0^+} = u, \end{cases}$$

then

$$-y^{1-2s}\partial_y U\big|_{y=0^+} = c_s L^s u$$

Explicit formulas.

$$U = \frac{y^{2s}}{4^s\Gamma(s)} \int_0^\infty e^{-y^2/(4\tau)} e^{-\tau L} u \, \frac{d\tau}{\tau^{1+s}} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-y^2/(4\tau)} e^{-\tau L} (L^s u) \, \frac{d\tau}{\tau^{1-s}}$$

If $L^s u = f$, clearly U solves

$$\begin{cases} -LU + \frac{1-2s}{y} U_y + U_{yy} = 0, & \text{for } y > 0, \\ -y^{1-2s}\partial_y U\big|_{y=0^+} = f, \end{cases}$$

and then

$$U\big|_{y=0^+} = c_s^{-1} L^{-s} f$$

- L: parabolic $\partial_t - a^{ij}(t)\partial_{ij}$ (with J. L. Torrea, 2016)
- L: linearized Monge–Ampère $\mathrm{tr}((D^2\varphi)^{-1}D^2)$ (with D. Maldonado, 2016)

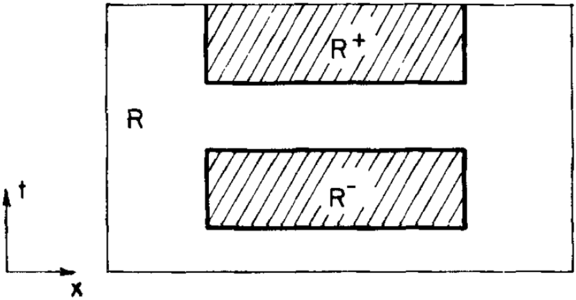
Application: parabolic Harnack inequality

Let $u(t,x)$ be a solution to

$$\begin{cases} (\partial_t - a^{ij}(t)\partial_{ij})^s u = 0, & \text{in } R = B_2 \times (0,2), \\ u(t,x) \geq 0, & \text{in } \mathbb{R}^n \times (-\infty, 2). \end{cases}$$

Then

$$\sup_{R^- = B_1 \times (\frac{1}{2},1)} u(t,x) \leq c \inf_{R^+ = B_1 \times (\frac{3}{2},2)} u(t,x)$$



Jürgen Moser's picture (1964)

Separated variables case

If u **depends only on** t then

$$(\partial_t - a^{ij}(t)\partial_{ij})^s u = \frac{1}{|\Gamma(-s)|}\int_0^\infty \frac{u(t)-u(t-\tau)}{\tau^{1+s}}\,d\tau$$
$$= \frac{1}{|\Gamma(-s)|}\int_{-\infty}^t \frac{u(t)-u(\tau)}{(t-\tau)^{1+s}}\,d\tau$$

- The fractional derivative of **Marchaud** (e.g. works by Allen–Caffarelli–Vasseur)

Separated variables case

If u **depends only on** t then

$$(\partial_t - a^{ij}(t)\partial_{ij})^s u = \frac{1}{|\Gamma(-s)|}\int_0^\infty \frac{u(t)-u(t-\tau)}{\tau^{1+s}}\,d\tau$$
$$= \frac{1}{|\Gamma(-s)|}\int_{-\infty}^t \frac{u(t)-u(\tau)}{(t-\tau)^{1+s}}\,d\tau$$

- The fractional derivative of **Marchaud** (e.g. works by Allen–Caffarelli–Vasseur)

Semigroup of translations *from the past*:

$$T_\tau u(t) = u(t-\tau) \qquad \tau > 0$$

Separated variables case

If u **depends only on** t then

$$(\partial_t - a^{ij}(t)\partial_{ij})^s u = \frac{1}{|\Gamma(-s)|}\int_0^\infty \frac{u(t)-u(t-\tau)}{\tau^{1+s}}\,d\tau$$
$$= \frac{1}{|\Gamma(-s)|}\int_{-\infty}^t \frac{u(t)-u(\tau)}{(t-\tau)^{1+s}}\,d\tau$$

- The fractional derivative of **Marchaud** (e.g. works by Allen–Caffarelli–Vasseur)

Semigroup of translations *from the past*:

$$T_\tau u(t) = u(t-\tau) \qquad \tau > 0$$

Infinitesimal generator:

$$\lim_{\tau\to 0^+} \frac{T_\tau u(t) - u(t)}{\tau} = -\lim_{\tau\to 0^+}\frac{u(t-\tau)-u(t)}{-\tau} = -D_{left}u(t)$$

Separated variables case

If u **depends only on** t then

$$(\partial_t - a^{ij}(t)\partial_{ij})^s u = \frac{1}{|\Gamma(-s)|}\int_0^\infty \frac{u(t)-u(t-\tau)}{\tau^{1+s}}\,d\tau$$
$$= \frac{1}{|\Gamma(-s)|}\int_{-\infty}^t \frac{u(t)-u(\tau)}{(t-\tau)^{1+s}}\,d\tau$$

- The fractional derivative of **Marchaud** (e.g. works by Allen–Caffarelli–Vasseur)

Semigroup of translations *from the past*:

$$T_\tau u(t) = u(t-\tau) \qquad \tau > 0$$

Infinitesimal generator:

$$\lim_{\tau\to0^+}\frac{T_\tau u(t)-u(t)}{\tau} = -\lim_{\tau\to0^+}\frac{u(t-\tau)-u(t)}{-\tau} = -D_{left}u(t)$$

Hence $T_\tau u(t) = e^{-\tau D_{left}}u(t)$ and

$$(D_{left})^s u(t) = \frac{1}{|\Gamma(-s)|}\int_0^\infty \frac{u(t)-u(t-\tau)}{\tau^{1+s}}\,d\tau \quad (\,=\textbf{Marchaud!})$$

Extension problem for Marchaud

Theorem (with Bernardis, Martín-Reyes and Torrea, *JDE* 2016)

If $U = U(t,y)$ solves

$$\begin{cases} -D_{left} U + \frac{1-2s}{y} U_y + U_{yy} = 0, & \text{for } t \in \mathbb{R},\ y > 0, \\ U(t,0) = u(t), & \text{for } t \in \mathbb{R}, \end{cases}$$

then

$$-y^{1-2s} \partial_y U(t,y)\big|_{y=0^+} = c_s (D_{left})^s u(t)$$

Explicit formulas. $\quad U(t,y) = \dfrac{y^{2s}}{4^s \Gamma(s)} \displaystyle\int_0^\infty e^{-y^2/(4\tau)} u(t-\tau) \frac{d\tau}{\tau^{1+s}}$

Extension problem for Marchaud

Theorem (with Bernardis, Martín-Reyes and Torrea, *JDE* 2016)

If $U = U(t,y)$ solves

$$\begin{cases} -D_{left}U + \frac{1-2s}{y}U_y + U_{yy} = 0, & \text{for } t \in \mathbb{R},\ y > 0, \\ U(t,0) = u(t), & \text{for } t \in \mathbb{R}, \end{cases}$$

then

$$-y^{1-2s}\partial_y U(t,y)\big|_{y=0^+} = c_s (D_{left})^s u(t)$$

Explicit formulas. $U(t,y) = \dfrac{y^{2s}}{4^s\Gamma(s)} \displaystyle\int_0^\infty e^{-y^2/(4\tau)} u(t-\tau)\, \frac{d\tau}{\tau^{1+s}}$

Corollary. Harnack inequality for Marchaud with time lag. There is $c > 0$ so that

$$\sup_{(1/4,1/2)} u \leq c \inf_{(3/4,1)} u,$$

for all u solution to

$$\begin{cases} (D_{left})^s u = 0, & \text{in } (0,1], \\ u \geq 0, & \text{in } (-\infty, 1]. \end{cases}$$

Thank you for your attention!