Universität Bielefeld

Nonlocal to local phase-transition for the Dirichlet problem

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Consider a sequence of nonlocal Dirichlet problems

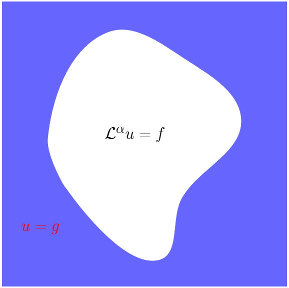
$$\mathcal{L}^\alpha u_\alpha = f \quad \text{in } \Omega \tag{1a}$$

$$u_\alpha = g \quad \text{on } \Omega^c, \tag{1b}$$

where $\mathcal{L}^\alpha$ is an integro-differential operator of the form

$$\mathcal{L}^\alpha u(x) = P.V. \int_{\mathbb{R}^d} (u(x) - u(y)) k^\alpha(x, y)\, \mathrm{d}y. \tag{2}$$

Model case $\mathcal{L}^\alpha = (-\Delta)^{\alpha/2}$



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GOAL: Prove that the family (u_{α}) of solutions to (1) converges to the solution u of a second order Dirichlet problem

$$-\operatorname{div}\left(a_{ij}(\cdot) \nabla u\right) u = f \quad \text{in } \Omega \tag{3a}$$

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TOOL: Γ-convergence

Assumptions on the kernels

Let $\alpha \in (0,2)$ and $k^\alpha : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty]$ be measurable. We assume

- Symmetry: $k^\alpha(x,y) = k^\alpha(y,x)$,
- Lévy condition: $\exists C_L > 0$:

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left(1 \wedge |x-y|^2\right) k^\alpha(x,y) \,\mathrm{d}y \le C_L \tag{4}$$

- Pointwise upper bound: $\exists \Lambda > 0$:

$$k^\alpha(x,y) \le \Lambda(2-\alpha) \, |x-y|^{-d-\alpha} \tag{5}$$

- Integrated lower bound: $\exists \lambda > 0$: $\forall u \in L^2(\mathbb{R}^d)$

$$\iint_{(\Omega^c \times \Omega^c)^c} (u(x)-u(y))^2 k_\alpha(x,y) \,\mathrm{d}x \,\mathrm{d}y \ge \lambda(2-\alpha) \iint_{(\Omega^c \times \Omega^c)^c} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}} \,\mathrm{d}x \,\mathrm{d}y \tag{6}$$

Definition

- For $0 < \alpha < 2$ the linear space $V^{\alpha/2}(\Omega, \mathbb{R}^d)$ is defined as

$$V^{\alpha/2}(\Omega, \mathbb{R}^d) = \left\{ f \in L^2(\mathbb{R}^d) \,\middle|\, \frac{f(x) - f(y)}{|x-y|^{d/2+s}} \in L^2((\Omega^c \times \Omega^c)^c) \right\}$$

and a norm on $V^{\alpha/2}(\Omega, \mathbb{R}^d)$ by

$$\|f\|^2_{V^{\alpha/2}(\Omega,\mathbb{R}^d)} = \|f\|^2_{L^2(\mathbb{R}^d)} + (2-\alpha) \iint_{(\Omega^c \times \Omega^c)^c} \frac{(f(x)-f(y))^2}{|x-y|^{d+2s}} \, \mathrm{d}y \, \mathrm{d}x$$

- $V_g^{\alpha/2}(\Omega, \mathbb{R}^d) = \left\{ f \in V^{\alpha/2}(\Omega, \mathbb{R}^d) | f = g \text{ a.e. on } \Omega^c \right\}$
- $L^2_{g,\Omega} = \left\{ f \in L^2(\mathbb{R}^d) | f = g \text{ a.e. on } \Omega^c \right\}$

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Note: $(\Omega^c \times \Omega^c)^c = (\Omega \times \Omega) \cup (\Omega \times \Omega^c) \cup (\Omega^c \times \Omega) = (\mathbb{R}^d \times \mathbb{R}^d) \setminus (\Omega^c \times \Omega^c)$.

Variational formulation

We say that u_α solves (1) in the variational sense, if $u_\alpha \in V_g^{\alpha/2}(\Omega, \mathbb{R}^d)$ and

$$\mathcal{E}^\alpha(u, v) = (f, v)_{L^2(\Omega)} \quad \text{for all } v \in V_0^{\alpha/2}(\Omega, \mathbb{R}^d)$$

where

$$\mathcal{E}^\alpha(u, v) = \frac{1}{2} \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} (u(x) - u(y)) (v(x) - v(y)) k^\alpha(x, y) \, \mathrm{d}y \, \mathrm{d}x$$

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Theorem ([FKV14])

Let $g \in V^{\alpha/2}(\Omega,\mathbb{R}^d)$ and let k^α satisfy the above assumptions. Then (1) *has a unique solution $u_\alpha \in V^{\alpha/2}(\Omega,\mathbb{R}^d)$.*

Nonlocal Dirichlet energy

Consider the functionals

$$F_g^\alpha(u) = \begin{cases} \mathcal{E}^\alpha(u,u) - (f,u)_{L^2(\Omega)}, & \text{if } u \in V_{g,\Omega}^{\alpha/2}(\mathbb{R}^d), \\ +\infty, & \text{else.} \end{cases}$$

$$F_g(u) = \begin{cases} \int_\Omega a_{ij}\partial_i u \partial_j u \,\mathrm{d}x - (f,u)_{L^2(\Omega)}, & \text{if } u - g \in H_0^1(\Omega), \\ +\infty, & \text{else.} \end{cases}$$

Lemma

If u is a minimizer of the functional F_g^α or F_g in $L^2(\mathbb{R}^d)$, then u is a variational solution of the Dirichlet-problem (1), (3) *respectively.*

Next: Prove that

$$F_g^\alpha \xrightarrow{\Gamma} F_g$$

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$$F(u) = \begin{cases} \int_\Omega a_{ij}\partial_i u \partial_j u \, \mathrm{d}x, & \text{if } u \in H^1(\Omega), \\ +\infty, & \text{else.} \end{cases}$$

Theorem

Let the family $(k_\alpha)_{\alpha\in(0,2)}$ satisfy the above assumptions uniformly in α. Then

$$\Gamma - \lim_{\alpha\to 2^-} F_\alpha = F \tag{7}$$

where the Γ-limit is taken with respect to the topology of $L^2(\mathbb{R}^d)$ and the entries of a_{ij} are given by

$$a_{ij}(x) = \lim_{\alpha\to 2^-} \int_0^1 \int_{S^{d-1}} \sigma_i \sigma_j k_\alpha(x, x+t\sigma)\, \mathrm{d}\sigma\, \mathrm{d}t. \tag{8}$$

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Related results:

- Aubert, Kornprobst [AK09]
- Mengesha, Du [MD15]

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$$\mathcal{E}(u,v) = \iint_{\Omega\times\Omega} \frac{((u(x)-u(y))^2}{|x-y|^{d+\alpha}}$$

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where the Γ-limit is taken with respect to the topology of $L^2(\mathbb{R}^d)$.

Proof: 2 Steps:

- $\liminf$-inequality: For all $u, (u_n) \in L^2(\mathbb{R}^d)$, s.t. $u_n \overset{n\to\infty}{\longrightarrow} u$

$$\liminf_{n\to\infty} F^{\alpha_n}(u) \geq F(u).$$

$\rightarrow$ mollification of u_n and u plus convexity and positivity of F^α

- $\limsup$-inequality: For all $u \in L^2(\mathbb{R}^d)$ $\exists (u_n) \in L^2(\mathbb{R}^d)$, s.t. $u_n \overset{n\to\infty}{\longrightarrow} u$ and

$$\limsup_{n\to\infty} F^{\alpha_n}(u) \leq F(u).$$

$\rightarrow$ constant sequence $u_n = u$ plus density argument.

Remark: [Pon04] For $u \in H^1(\Omega)$

$$\int_\Omega \int_\Omega (u(x) - u(y))^2 \, k^\alpha(x,y) \, \mathrm{d}y \, \mathrm{d}x \overset{\alpha \to 2^-}{\longrightarrow} F(u)$$

pointwise and also in the sense of $\Gamma-$convergence. This is different in our setting.

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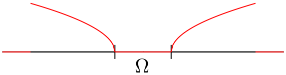
Example: Let $\Omega = (-1, 1)$ and $k^\alpha(x,y) = (2-\alpha)\,|x-y|^{-d-\alpha}$. Consider the function

$$f(x) = \begin{cases} (|x|-1)^{1/2}, & \text{if } 1 < |x| < 2 \\ 0, & \text{else.} \end{cases}$$

Then $u \in H^1(\Omega)$, $u \in V^{\alpha/2}(\Omega, \mathbb{R}^d)$ for all $\alpha \in (0,2)$
but

$$\lim_{\alpha \to 2^-} F^\alpha(u) = 2 \neq 0 = F(u).$$



Our theorem above: $\Gamma - \lim_{\alpha \to 2^-} F^\alpha(u) = F(u)$.

So far: Free engery Γ-converges.

Left: Prove that $F_g^\alpha(u) \xrightarrow{\Gamma} F_g(u)$:

- $\liminf$ $-$inequality $\surd$
- $\limsup$ $-$inequality: Find a recovery sequence in $L^2_{g,\Omega}(\mathbb{R}^d)$.

Idea: For $u \in L^2_{g,\Omega}$ there is a sequence $(u_n) \in L^2(\mathbb{R}^d)$, such that the $\limsup$-inequality holds. Let $\nu \in \mathbb{N}$ large enough, set

$$v_n = \frac{1}{\nu} \sum_{i=1}^{\nu} (1 - \varphi_i) u + \varphi_i u_n$$

where φ_i, $i = 1, .., \nu$ are suitable cut-off functions with $\operatorname{supp}\{\varphi_i\} \subset K_i \Subset \Omega$. Then $v_n \in L^2_{g,\Omega}(\mathbb{R}^d)$ and satisfies the $\limsup$ $-$inequality.

Theorem

Let k_α, $\alpha \in (0,2)$ *satisfy the above assumptions uniformly and let* $g \in \bigcap_{\alpha<2} V^{\alpha/2}(\Omega, \mathbb{R}^d)$, *then the solution* u_α *of*

$$\mathcal{L}^\alpha u_\alpha = f \qquad \text{in } \Omega$$
$$u_\alpha = g \qquad \text{on } \Omega^c,$$

converge to the solution u *of*

$$\mathrm{div}\,(a_{ij}(\cdot)\nabla u)\, u = f \qquad \text{in } \Omega$$
$$u = g \qquad \text{on } \partial\Omega,$$

in $L^2(\mathbb{R}^d)$ *as* $\alpha \to 2^-$.

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