Long-time behavior in a doubly nonlocal Fisher–KPP equation

Pasha Tkachov

1 Basic equation

Aim: study of a long-time behavior of the following evolution equation

$$\frac{\partial u}{\partial t} = \varkappa^+(a^+ * u) - mu - \varkappa^- u(a^- * u). \qquad (1)$$

Classical solution: $u \in C(\mathbb{R}_+ \to E) \cap C^1((0,\infty) \to E)$.
Space: $E = BUC(\mathbb{R}^d)$ or $E = L^\infty(\mathbb{R}^d)$.
Dispersion and competition kernels: $0 \le a^\pm \in L^1(\mathbb{R}^d)$, with constant rates $\varkappa^\pm > 0$.
Mortality rate (constant): $m > 0$.

$$\int_{\mathbb{R}^d} a^+(y)dy = \int_{\mathbb{R}^d} a^-(y)dy = 1, \quad (a^\pm * u)(x,t) := \int_{\mathbb{R}^d} a^\pm(x-y)u(y,t)dy.$$

Initial condition: $u(x,0) = u_0(x),\ x \in \mathbb{R}^d$.
Constant solutions: $u \equiv 0$, $u \equiv \theta := \dfrac{\varkappa^+ - m}{\varkappa^-}$.

Theorem 1 (Existence and uniqueness). *Let $u_0 \in E$ and $u_0 \ge 0$. Then, for any $T > 0$, there exists a unique solution $u \ge 0$ to the equation (1) in E, such that $u \in C([0,T], E) \cap C^1((0,T], E)$.*

Theorem 2 (Global boundedness). *Let a^- be separated from zero at the origin and a^+ has a regular behavior at ∞, for example,*

$$a^+(x) \le \frac{A}{1 + |x|^{d+\varepsilon}}, \qquad A, \varepsilon > 0.$$

Then $0 \le u_0 \in BUC(\mathbb{R}^d)$ implies

$$\sup_{t\ge 0} \sup_{x \in \mathbb{R}^d} u(x,t) < \infty.$$

Note that if $a^- = \frac{\delta_{-1}+\delta_1}{2}$, then there exist bounded u_0 and compactly supported a^+ such that $\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \to \infty$, as $t \to \infty$.

From now, we will suppose that

$$\varkappa^+ > m, \qquad (A1)$$
$$\varkappa^+ a^+(x) \ge (\varkappa^+ - m)a^-(x). \qquad (A2)$$

If (A1) fails, then the solution tends uniformly to zero. In particular the constant solution θ is positive if and only if (A1) holds.
The assumption (A2) yields a stability of θ and provides a comparison principle:

Theorem 3 (Comparison principle).

- $0 \le u_0 \le \theta = \frac{\varkappa^+ - m}{\varkappa^-}$ *implies* $0 \le u(x,t) \le \theta,\ t \ge 0,\ x \in \mathbb{R}^d$,
- $0 \le u_0 \le v_0 \le \theta$ *implies* $u(x,t) \le v(x,t),\ t \ge 0,\ x \in \mathbb{R}^d$.

2 History of derivation

- 'Crabgrass model' on $\mathbb{Z}^d$, for $\varkappa^+ = \varkappa^-$, $a^+ = a^-$
  `Durrett'88 (Bull. AMS)`
- (Heuristic) A Spatial Ecology model on $\mathbb{R}^d$
  `Bolker/Pacala'97 (Amer. Naturalist)`
- Stochastic approach, finite systems in $\mathbb{R}^d$, $E = L^1(\mathbb{R}^d)$
  `Fournier/Méléard'04 (Ann. Appl. Prob.)`
- Semigroup approach for the derivation of kinetic equations, infinite systems in $\mathbb{R}^d$, $E = L^\infty(\mathbb{R}^d)$ + some conditions
  `Finkelshtein/Kondratiev/Kutoviy'12 (J. Funct. Anal.)`
- Evolution in a scale of L^∞-spaces, weaker conditions
  `Finkelshtein/Kondratiev/Kozitsky/Kutoviy'15 (Math. Models & Meth. Appl. Sci.)`

3 Similar equations

$$\frac{\partial u}{\partial t}(x,t) = \alpha \Delta u(x,t) + f(u(x,t)), \qquad \text{(F-KPP)}$$

where $f(u) = ku(1-u)$, $k > 0$, is a particular case.
`Fisher'37, Kolmogorov/Petrovsky/Piskunov'37, Aronson/Weinberger'78`

$$\frac{\partial u}{\partial t}(x,t) = \varkappa^+((a^+ * u)(x,t) - u(x,t)) + f(u(x,t)).$$

`Schumacher'80, Coville/Dupaigne'05,'07, Coville/Dávila/Martínez'08, Yagisita'09, Li/Sun/Wang'10, Garnier'11, Sun/Li/Wang'11, Aguerrea/Gomez/Trofimchuk'12, Bonnefon/Coville/Garnier/Roques'14`

$$\frac{\partial u}{\partial t}(x,t) = \Delta u(x,t) + F(u(x,t), (a^- * u)(x,t)),$$

where $F(u,v) = ku(1-v)$, $k > 0$, is a particular case.
`Gourley'00, Genieys/Volpert/Auger'06, Berestycki/Nadin/Perthame/Ryzhik'09, Apreutesei/Bessonov/Volpert/Vougalter'10, Nadin/Perthame/Tang'11, Fang/Zhao'11, Alfaro/Coville'12, Hamel/Ryzhik'14, Achleitner/Kuehn'15, Faye/Holzer'15`

4 Constant speed of propagation

For details see:
`Finkelshtein/Kondratiev/T.'15 (ArXiv:1508.02215)`

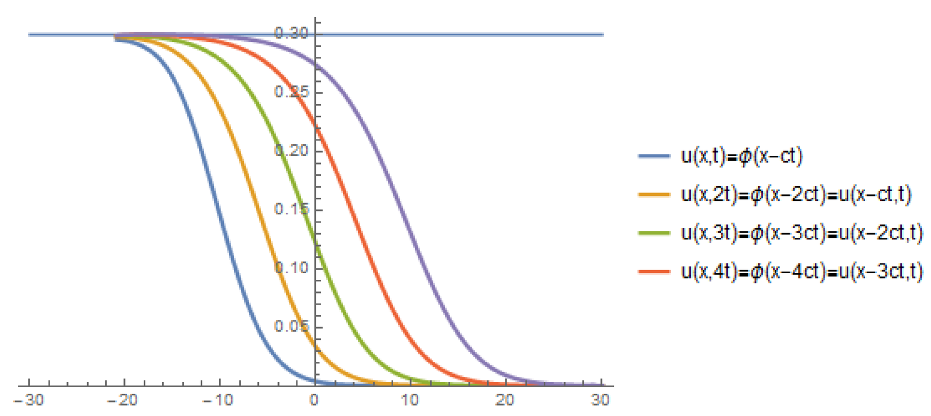
Let $\xi \in S^{d-1}$ be fixed and there exist $\lambda > 0$ such that

$$\int_{\mathbb{R}^d} a^+(x)e^{\lambda x\cdot\xi}dx < \infty. \qquad \text{(A3,a)}$$

The assumption states that a^+ decays fast in a direction ξ (light-tailed).

Definition 4. *A solution u to the equation (1) is said to be a traveling wave solution with a speed $c \in \mathbb{R}$ and in a direction ξ if and only if there exists a profile $\phi \in \mathcal{M}_\theta(\mathbb{R})$ (the set of all decreasing and right-continuous functions $f : \mathbb{R} \to [0,\theta]$), such that*

$$\phi(-\infty) = \theta, \qquad \phi(+\infty) = 0,$$
$$u(x,t) = \phi(x\cdot\xi - ct), \quad t \ge 0, \text{ a.a. } x \in \mathbb{R}.$$



$$c_*(\xi) = \inf_{\lambda>0} G(\lambda) := \inf_{\lambda>0} \frac{1}{\lambda}\Big(\varkappa^+ \int_{\mathbb{R}^d} a^+(x)e^{\lambda x\cdot\xi}dx - m\Big). \qquad (2)$$

Theorem 5. *For any $c \ge c_*(\xi)$, there exists a unique traveling wave solution to (1) with a profile $\phi \in \mathcal{M}_\theta(\mathbb{R})$ and the speed c. For any $c < c_*(\xi)$, such a traveling wave solution does not exist.*

- Since a^+ is not supposed to be symmetric, there exists a^+ such that
  $$c_*(\xi) < 0.$$
- If $c \ne 0$, the profile $\phi \in C_b^\infty(\mathbb{R})$, and $\phi \in C(\mathbb{R})$ otherwise.
- Let λ_0 be such that $c = G(\lambda_0)$. For some $D > 0$,
  $$\phi(t) \sim Dt^{j-1}e^{-\lambda_0 t}, t \to \infty,$$
  where $j = 1$, for $c > c_*$. Both cases $j = 1$ and $j = 2$ are possible if $c = c_*$.

Let, for some $\lambda > 0$, (A3,a) holds for all $\xi \in S^{d-1}$.
We define $\Gamma := \{x \in \mathbb{R}^d \mid x\cdot\xi \le c_*(\xi),$ *for all* $\xi \in S^{d-1}\}$. (3)

Theorem 6. *Let u_0 be such that, for all $\lambda > 0$,*

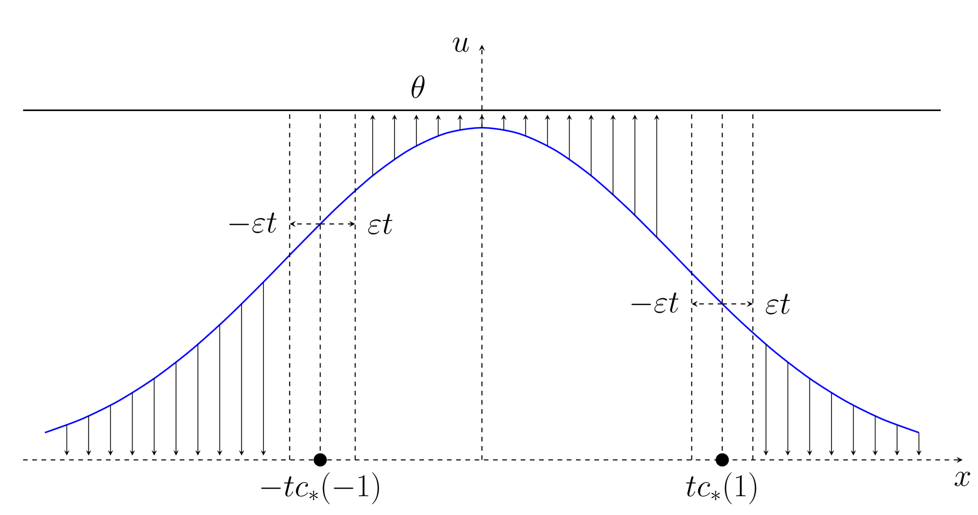
$$\operatorname*{esssup}_{x\in\mathbb{R}^d} u_0(x)e^{\lambda|x|} < \infty,$$

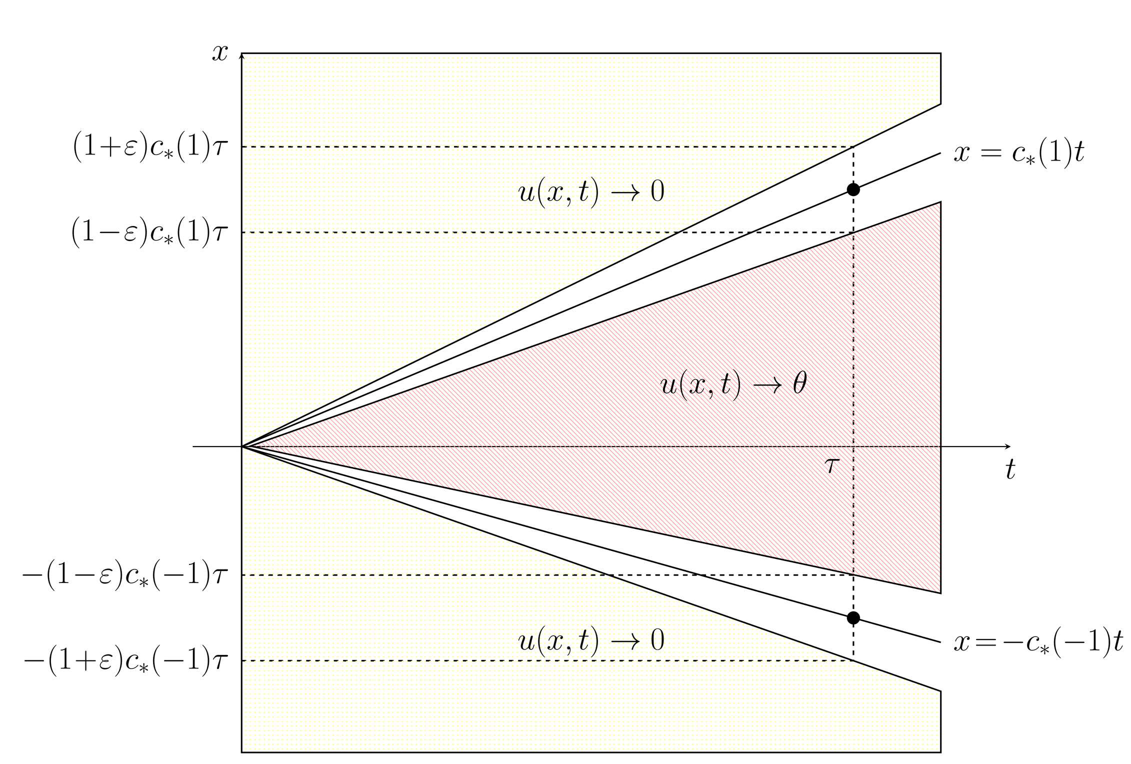
and let u be the corresponding solution to (1). Then, for any $\varepsilon > 0$,

$$\lim_{t\to\infty} \operatorname*{esssup}_{x \notin (1+\varepsilon)t\Gamma} u(x,t) = 0.$$

Let $u_0 \not\equiv 0$. Then, for any $\varepsilon \in (0,1)$,

$$\lim_{t\to\infty} \operatorname*{essinf}_{x \in (1-\varepsilon)t\Gamma} u(x,t) = \theta.$$





5 Acceleration

`(Paper in preparation)`

In the previous section both a^+ and u_0 were fast-decaying, which provides a finite speed of propagation of the solution. On the other hand if one of the functions is slowly decaying an acceleration appears. Let the following assumption holds, for any $\lambda > 0$,

$$\int_{\mathbb{R}^d} (a^+ * u_0)(x)e^{\lambda|x|}dx = \infty. \qquad \text{(A3,b)}$$

In order to get asymptotic results, it is necessary to introduce functions with a regular decay.
We say that a bounded integrable function $b : \mathbb{R} \to \mathbb{R}_+$ is long-tailed if

$$b(s+r) \sim b(s), \ s \to \infty, \ r > 0.$$

A bounded integrable function $c : \mathbb{R} \to \mathbb{R}_+$ is sub-exponential on $\mathbb{R}$ if it is long-tailed and

$$(c * c)(s) \sim 2c(s), \ s \to \infty.$$

For a deeper discussion of the definitions we refer on the books by
`Borovkov/Borovkov'08`
`Foss/Korshunov/Zachary'11`

Theorem 7 (Inside the front). *Let $0 \le u_0 \le \theta$ be separated form zero at the origin and $b(s)$ be long-tailed, such that, for large s, $b(s)$ is decreasing and $\log b(s)$ is convex. Suppose that*

$$(a^+ * u_0)(x) \ge b(|x|), x \in \mathbb{R}^d.$$

Then, for any small $\varepsilon > 0$,

$$\lim_{t\to\infty} \operatorname*{essinf}_{|x| \le \eta((1-\varepsilon)t,b)} u(x,t) = \theta, \qquad (4)$$

where $\eta(s,b) = b^{-1}(e^{(m-\varkappa^+)s})$.

Theorem 8 (Outside the front). *Suppose that c is sub-exponential on $\mathbb{R}$ such that, for large s, $c(s)$ is decreasing and $\log c(s)$ is convex. Let $0 \le u_0 \le \theta$ and a^+ be such that,*

$$\operatorname*{esssup}_{x\in\mathbb{R}^d} \frac{u_0(x)}{c(|x|)} < \infty, \qquad \lim_{|x|\to\infty} \frac{a^+(x)|x|^{d-1}}{c(|x|)} = 0.$$

Then for any small $\varepsilon > 0$,

$$\lim_{t\to\infty} \operatorname*{esssup}_{|x| \ge \eta((1+\varepsilon)t,c)} u(x,t) = 0, \qquad (5)$$

where $\eta(s,c) = c^{-1}(e^{(m-\varkappa^+)s})$

In the following examples $\eta(t) := \eta(t,b) \sim \eta(t,c)$, so that both (4) and (5) hold, for the same $\eta(t)$. Moreover $\eta(t) \sim \eta(t, a^+ * u_0)$.
In particular if $u(\rho(t),t) = \lambda \in (0,\theta)$, then

$$\rho(t) \in \Big(\eta\big((1-\varepsilon)t\big), \eta\big((1+\varepsilon)t\big)\Big).$$

Example 1. $u_0(x) = o(a^+(x)) \ \Rightarrow\ (a^+ * u_0)(x) \sim a^+(x) =: b(x)$

Then for $|x| \ge R > 0$ and t sufficiently large,

1. $a^+(x) = e^{-|x|^\gamma}$; $\qquad \eta(t) \sim \big((\varkappa^+ - m)t\big)^{\frac{1}{\gamma}}$,
2. $a^+(x) = \dfrac{1}{(1+|x|^2)^p}$; $\qquad \eta(t) \sim \exp\big(\frac{\varkappa^+ - m}{2p}t\big)$,
3. $a^+(x) = \exp(-\dfrac{(\log|x|)^2}{2\sigma^2})$; $\qquad \eta(t) \sim \exp\big(\sigma\sqrt{2t}\big)$.

Example 2. $a^+(x) = o(u_0(x)) \ \Rightarrow\ (a^+ * u_0)(x) \sim u_0(x) =: b(x)$

If u_0 satisfies one of the cases 1-3 instead of a^+, then $\eta(t)$ remains the same.