Spatial asymptotics at infinity for heat kernels of some nonlocal operators

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(joint work with Kamil Kaleta)

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Let $d \in \{1, 2, \dots\}$, $b \in \mathbb{R}^d$, and $\nu(dy) = \nu(y)\,dy$ be an absolutely continuous Lévy measure on $\mathbb{R}^d \setminus \{0\}$, i.e.,

$$\int_{\mathbb{R}^d \setminus \{0\}} \left(1 \wedge |y|^2\right) \nu(y)\,dy < \infty.$$

Let $A = (a_{jk})$ be a symmetric nonnegative definite matrix.
We consider the convolution semigroup of probability measures $\{P_t,\, t \geqslant 0\}$ with the Fourier transform $\mathcal{F}(P_t)(\xi) = \exp(-t\Phi(\xi))$, where

$$\Phi(\xi) = -i\xi \cdot b + \xi \cdot A\xi + \int \left(1 - e^{i\xi \cdot y} + i\xi \cdot y \mathbf{1}_{B(0,1)}(y)\right) \nu(y)\,dy, \quad \xi \in \mathbb{R}^d.$$

We have $\frac{\partial P_t f(x)}{\partial t} = L P_t f(x)$, $P_0 f(x) = f(x)$, where

$$Lf(x) = b \cdot \nabla f(x) + \sum_{j,k=1}^{d} a_{jk} \frac{\partial^2 f}{\partial x_j \partial x_k}(x) + \int_{\mathbb{R}^d \setminus \{0\}} \left(f(x+u) - f(x) - \nabla f(x) \cdot u 1_{B(0,1)}(u)\right) \nu(du),$$

for $f \in C_b^2(\mathbb{R}^d)$.

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We have $\frac{\partial P_t f(x)}{\partial t} = LP_t f(x)$, $P_0 f(x) = f(x)$, where

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Rotation invariant stable semigroup

Let $\nu(dh) = c_{\alpha,d}|h|^{-d-\alpha}dh$, $\alpha \in (0,2)$, $b = 0$, $A = 0$.

$$\Phi(\xi) = |\xi|^\alpha, \quad \xi \in \mathbb{R}^d, \quad L = -(-\Delta)^{\alpha/2}.$$

There exists a smooth density p_t of P_t and

$$\lim_{|x|\to\infty} |x|^{d+\alpha} p_1(|x|) = c_{\alpha,d},$$

(see Polya 1923, Blumenthal, Getoor 1960), i.e.

$$\lim_{|x|\to\infty} \frac{p_1(|x|)}{\nu(|x|)} = 1,$$

and by scaling $p(t,x) = t^{-d/\alpha} p(1, t^{-1/\alpha}x)$ we get

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Assumptions

(A) $A \equiv 0$ or $\inf_{|\xi|=1} \xi \cdot A\xi > 0$

(B) Denote

$$\Phi_\nu(\xi) := \Phi(\xi) - \xi \cdot A\xi \quad \text{and} \quad \Psi(r) = \sup_{|\xi|\leqslant r} \operatorname{Re} \Phi_\nu(\xi), \quad r > 0,$$

and

$$h(t) := \frac{1}{\Psi^{-1}\left(\frac{1}{t}\right)}, \quad t > 0.$$

There exist a nonempty and bounded set $T \subset (0, \infty)$ and a constant $C > 0$ such that

$$\int_{\mathbb{R}^d} e^{-t \operatorname{Re} \Phi_\nu(\xi)} |\xi| d\xi \leqslant C[h(t)]^{-d-1}, \quad t \in T$$

(sasisfied if, e.g., if $\Phi_\nu(\xi) \asymp F(|\xi|)$ for some increasing, continuous function F such that $F^{-1}(2s) \leqslant cF^{-1}(s)$, $s > 0$).

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(C) $\nu(dx) = \nu(x)dx$ and there exists a nonincreasing function $f : (0,\infty) \to (0,\infty)$ such that

$$\nu(x) \asymp f(|x|), \quad x \in \mathbb{R}^d \setminus \{0\},$$

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$$G(r) := \sup_{|x|>1} \frac{\int_{\substack{|x-y|>r \\ |y|>r}} f(|x-y|) f(|y|) dy}{f(|x|)} \searrow 0 \qquad \text{as} \quad r \to \infty.$$

(D) There exist $\Theta \subset \mathbb{S}$ and $\kappa \geqslant 0$ such that

$$\lim_{r\to\infty} \frac{\nu(r\theta - y)}{\nu(r\theta)} = e^{\kappa(\theta \cdot y)}, \quad y \in \mathbb{R}^d, \theta \in \Theta.$$

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Main result

Theorem 1

*Let the assumptions **(A)**, **(B)**, **(C)** and **(D)** hold. Then, for every $t \in T$, $\theta \in \Theta$ and $y \in \mathbb{R}^d$,*

$$\lim_{r\to\infty} \frac{p_t(r\theta - y)}{t\,\nu(r\theta)} = e^{t\widetilde{\Phi}(\kappa\theta)+\kappa\theta\cdot y}, \tag{1}$$

where

$$\widetilde{\Phi}(\xi) = \xi\cdot b + \xi\cdot A\xi + \int_{\mathbb{R}^d\setminus\{0\}} \left(e^{\xi\cdot y} - 1 - \xi\cdot y\mathbf{1}_{B(0,1)}(y)\right)\nu(y)\,dy.$$

Moreover, if the convergence in (D) is uniform on Θ then (1) is uniform on Θ and if the convergence in (D) is uniform with respect to (θ, y) on $\Theta\times B(0,\varrho)$ for every $\varrho > 0$ then (1) is uniform in (t,θ,y) on each $T\times\Theta\times B(0,\varrho)$, $\varrho > 0$.

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Finite Lévy measures

Let $\nu(\mathbb{R}^d) < \infty$ and let p_t be the density of the absolutely continuous part of P_t.

Theorem 2

*Let the assumptions **(A)**, **(C)** and **(D)** hold. Then, for every $t \in T$, $\theta \in \Theta$ and $y \in \mathbb{R}^d$,*

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Moreover, if the convergence in (D) is uniform on Θ then (2) is uniform on Θ and if the convergence in (D) is uniform with respect to (θ, y) on $\Theta\times B(0,\varrho)$ for every $\varrho > 0$ then (2) is uniform in (t,θ,y) on each $T\times\Theta\times B(0,\varrho)$, $\varrho > 0$.

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Stable semigroups (with drift)

Let $A = 0$, $b \in \mathbb{R}^d$, $\nu(x) = g(x/|x|)|x|^{-d-\alpha}$, where $\alpha \in (0,2)$ and $c^{-1} \leqslant g(\theta) \leqslant c$, for $\theta \in \mathbf{S}$.

Assumption (C): $\nu(x) \asymp f(x) = |x|^{-\alpha-d}$ and

$$G(r) := \sup_{|x|>1} \frac{\int_{\substack{|x-y|>r \\ |y|>r}} f(|x-y|)f(|y|)dy}{f(|x|)} \asymp r^{-\alpha} \searrow 0$$

Assumption (D) with $\kappa = 0$:

$$\lim_{r\to\infty} \frac{\nu(r\theta - y)}{\nu(r\theta)} = \lim_{r\to\infty} \frac{|\theta - y/r|^{-\alpha-d} g((\theta - y/r)/|\theta - y/r|)}{g(\theta)} = 1,$$

if only g is continuous in $\theta \in \mathbf{S}$ (uniformly on $\Theta \subset \mathbf{S}$). We obtain (see also [Dziubański 1991])

$$\lim_{r\to\infty} \frac{p_t(r\theta - y)}{t\, r^{-\alpha-d}} = g(\theta),$$

uniformly in (t, θ, y) on each $(0, t_0) \times \Theta \times B(0, \rho)$, $t_0 > 0$, $\rho > 0$.

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(see also [Kaleta, Lorinczi 2016]).

Lemma 3 (Kaleta, S 2015)

*If $f(s) = s^{-d-\alpha}(1+s)^{-\eta}e^{-ms^{\beta}}$, then (C) holds **if and only if***

(a) $m = 0$ *and* $\eta > -\alpha$ *or*

(b) $m > 0$, $\beta \in (0,1)$ *and* $\eta \geqslant -d-\alpha$ *or*

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(c) $m > 0$, $\beta = 1$ *and* $\eta > -(\frac{d-1}{2}+\alpha)$.

On assumption (C)

$\nu(dx) = \nu(x)dx$ and there exists a nonincreasing function $f : (0, \infty) \to (0, \infty)$ such that

$$\nu(x) \asymp f(|x|), \quad x \in \mathbb{R}^d \setminus \{0\},$$

and

$$G(r) := \sup_{|x|>1} \frac{\int_{\substack{|x-y|>r \\ |y|>r}} f(|x-y|) f(|y|) dy}{f(|x|)} \searrow 0 \qquad \text{as} \quad r \to \infty.$$

(see also [Kaleta, Lorinczi 2016]).

Lemma 3 (Kaleta, S 2015)

*If $f(s) = s^{-d-\alpha}(1+s)^{-\eta} e^{-ms^{\beta}}$, then (C) holds **if and only if***

(a) $m = 0$ *and* $\eta > -\alpha$ *or*

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Lemma 4 (Kaleta, S 2016)

If additionally $\nu(x) = \nu(-x)$ $(A = 0,\ b = 0)$ then

$$c_1 \leqslant \frac{p_t(x)}{t f(|x|)} \leqslant c_2, \quad |x| > R, t \in (0, t_0),$$

for some $R > 0$, t_0, $c_1, c_2 > 0$ ***if and only if*** *one of (a), (b) or (c) holds.*

Assumption (C)

$\nu(dx) = \nu(x)dx$ and there exists a nonincreasing function $f : (0, \infty) \to (0, \infty)$ such that

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Lemma 4 (Kaleta, S 2016)

If additionally $\nu(x) = \nu(-x)$ ($A = 0$, $b = 0$) then

$$c_1 \leqslant \frac{p_t(x)}{tf(|x|)} \leqslant c_2, \quad |x| > R, t \in (0, t_0),$$

*for some $R > 0$, t_0, $c_1, c_2 > 0$ **if and only if** one of (a), (b) or (c) holds.*

Relativistic stable semigroup

Taking $b = 0$, $A = 0$ and

$$\nu(x) = \frac{c}{|x|^{d+\alpha}} e^{-m^{1/\alpha}|x|} \varphi(m^{1/\alpha}|x|),$$

where

$$\varphi(\xi) = \int_0^\infty e^{-v} v^p (\xi + v/2)^p \, dv, \quad \xi \geqslant 0, p = \frac{d+\alpha-1}{2}.$$

We have

$$\nu(x) \asymp |x|^{-d-\alpha}(1+|x|)^{\frac{d+\alpha-1}{2}} e^{-m^{1/\alpha}|x|}, \ \ L = -((-\Delta + m^{2/\alpha})^{\alpha/2} - m),$$

and

$$\lim_{r\to\infty} \frac{\nu(r\theta - y)}{\nu(r\theta)} = e^{m^{1/\alpha}\theta\cdot y},$$

so

$$\lim_{r\to\infty} \frac{p_t(r\theta - y)}{t\nu(r\theta)} = e^{mt + m^{1/\alpha}\theta\cdot y}, \ \theta \in \mathbf{S}, t \in (0, t_0).$$

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Tempered stable semigroup 1

Let $A=0$, $b=0$, and

$$\nu(x) = g(x/|x|)f(|x|),$$

where

$$f(s) = s^{-d-\alpha}(1+s)^{-\eta}e^{-ms^{\beta}}, \quad s>0,$$

$g : \mathbf{S} \to (0,\infty)$ is continuous and $c^{-1} \leqslant g(\theta) \leqslant c$, $\alpha \in (0,2)$, $\beta \in (0,1)$, $\eta > -d-\alpha$.

Then we have

$$\lim_{r\to\infty} \frac{\nu(r\theta - y)}{\nu(r\theta)} = 1,$$

and we get

$$\lim_{r\to\infty} \frac{p_t(r\theta - y)}{t\nu(r\theta)} = 1,$$

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$g : \mathbf{S} \to (0, \infty)$ is continuous and $c^{-1} \leqslant g(\theta) \leqslant c$, $\alpha \in (0, 2)$, $\eta > -\alpha - \frac{d-1}{2}$.

Then we have

$$\lim_{r\to\infty} \frac{\nu(r\theta - y)}{\nu(r\theta)} = e^{m\theta\cdot y},$$

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Beyond (C)

Let $A = 0$, $b = 0$,

$$\nu(x) \asymp |x|^{-\alpha-d}(1+|x|)^{-\eta}e^{-m|x|}, \quad x \in \mathbb{R}^d,$$

where $m > 0$, $\eta \in [-\alpha - d, -\alpha - \frac{d-1}{2}]$.
Then we have [S. 2011, Kaleta, S. 2015]

$$c_1 \min\{t^{-d/\alpha}, t|x|^{-\alpha-d}(1+|x|)^{-\eta}e^{-m|x|}\} \leqslant$$

$$p_t(x) \leqslant c_2 \min\{t^{-d/\alpha}, t|x|^{-\alpha-d}(1+|x|)^{-\eta}e^{-m|x|/4}\}, \quad t \in (0, t_0).$$

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$$M_1 (R/r)^{\beta_1} \leqslant \frac{f(r)}{f(R)} \leqslant M_2 (R/r)^{\beta_2}, \quad 1 > R > r > 0,$$

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$$p_t(x) \asymp h(t)^{-d}, \quad |x| \leqslant \eta_* h(t),$$

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see [S 2016, Chen, Kim, Kumagai 2008].

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Thank you very much for your attention!