Study of a family of higher order nonlocal degenerate parabolic equations

Rana Tarhini

Université Paris Est, Laboratoire d'Analyse et de Mathématiques Appliquées, UMR 8050
61 avenue du Général de Gaulle, 94010 Créteil, France
rana.tarhini@math.cnrs.fr, rana.tarhini@univ-paris-est.fr

1. Introduction

In this poster, we study the following problem

$$\begin{cases} \partial_t u + \partial_x(u^n \partial_x I(u)) = 0 & \text{for } x \in \Omega, \quad t > 0, \\ \partial_x u = 0, u^n \partial_x I(u) = 0 & \text{for } x \in \partial\Omega, \quad t > 0, \\ u(0,x) = u_0(x) & \text{for } x \in \Omega, \end{cases} \qquad (1)$$

where $\Omega = (a,b)$ is a bounded interval in $\mathbb{R}$, n is a positive real number and I is a nonlocal elliptic negative operator of order α defined as the $\alpha/2$ power of the Laplace operator with Neumann boundary conditions $I = -(-\Delta)^{\frac{\alpha}{2}}$ where $\alpha \in (0,2)$. The equation under consideration is a nonlocal degenerate parabolic equation of order $\alpha + 2$.
The case $\alpha = 1$ was studied by Imbert and Mellet [1] who proved the existence of nonnegative solutions constructed by passing to the limit in a regularized problem for nonnegative initial data with appropriate conditions.
We will generalize the result of [1] to the cases $0 < \alpha < 2$.
In the case $\alpha > 1$ we get the local uniform convergence of approximate solutions due to the following embedding in dimension 1

$$H^{\frac{\alpha}{2}}(\Omega) \hookrightarrow C^{0,\frac{\alpha-1}{2}}(\Omega).$$

This convergence allows one to pass to the limit in the nonlinear term and then allows us to construct nonnegative solutions for nonnegative initial data merely in $H^{\frac{\alpha}{2}}(\Omega)$.
In the case $\alpha < 1$ because of the following embedding

$$H^{\frac{\alpha}{2}}(\Omega) \hookrightarrow L^p(\Omega) \text{ for all } p < \frac{2}{1-\alpha},$$

we can get a compactness result in $L^p(\Omega)$ only for $p < \frac{2}{1-\alpha}$ and not for all $p < \infty$ as in the case $\alpha = 1$. Neverthless, we recover a compactness result for the term $I(u)$ which allows us to pass to the limit and conclude.

2. Main results

Theorem 1 (Existence of solutions for $0 < \alpha \leqslant 1$) *Let $n \geqslant 1$ and $\alpha \in (0,1]$. For any nonnegative initial condition $u_0 \in H^{\frac{\alpha}{2}}(\Omega)$ such that*

$$\int_\Omega G(u_0)dx < \infty$$

where G is a nonnegative function such that $G''(s) = \frac{1}{s^n}$, there exists a nonnegative function

$$u \in L^\infty(0,T; H^{\frac{\alpha}{2}}(\Omega)) \cap L^2(0,T; H_N^{\frac{\alpha}{2}+1}(\Omega))$$

which satisfies on $Q = (0,T) \times \Omega$

$$\iint_Q u\partial_t\varphi dtdx - \iint_Q nu^{n-1}\partial_x u I(u)\partial_x\varphi dxdt - \iint_Q u^n I(u)\partial^2_{xx}\varphi dxdt = -\int_\Omega u_0\varphi(0,.)dx$$

for all $\varphi \in \mathcal{D}([0,T) \times \bar{\Omega})$ satisfying $\partial_x\varphi = 0$ on $(0,T) \times \partial\Omega$.

Theorem 2 (Existence of solutions for $1 < \alpha < 2$) *Let $n \geqslant 1$ and $\alpha > 1$. For any nonnegative initial condition $u_0 \in H^{\frac{\alpha}{2}}(\Omega)$, there exists a nonnegative function*

$$u \in C_{t,x}^{\frac{\alpha-1}{2(\alpha+2)},\frac{\alpha-1}{2}}(Q)$$

such that

$$\partial_x I(u) \in L^2_{loc}(Q_+)$$

and that satisfies

$$\iint_Q u\partial_t\varphi dtdx + \iint_{Q_+} u^n\partial_x I(u)\partial_x\varphi dxdt = -\int_\Omega u_0\varphi(0,.)dx$$

where $Q_+ = \{u > 0\} \cap Q$, for all $\varphi \in \mathcal{D}([0,T) \times \bar{\Omega})$ satisfying $\partial_x\varphi = 0$ on $(0,T) \times \partial\Omega$.

3. Operator I

Spectral definition.

We define the operator I by

$$I : \sum_{k=0}^{\infty} c_k\varphi_k \longrightarrow -\sum_{k=0}^{\infty} c_k\lambda_k^{\frac{\alpha}{2}}\varphi_k \quad \text{which maps } H_N^\alpha(\Omega) \text{ onto } L^2(\Omega)$$

where $\{\lambda_k, \varphi_k\}_{k\geq 0}$ are the eigenvalues and corresponding eigenvectors of the Laplacian operator in Ω with Neumann boundary conditions on $\partial\Omega$.

Integral representation.

The operator I can also be represented as a singular integral operator, for a smooth function $u : \Omega \to \mathbb{R}$ for all $x \in \Omega$,

$$I(u)(x) = \int_\Omega (u(y) - u(x))K(x,y)dy$$

where $K(x,y)$ is defined as follows. For all $x, y \in \Omega$

$$K(x,y) = c_\alpha \sum_{k\in\mathbb{Z}} \left(\frac{1}{\mid x - y - 2k \mid^{1+\alpha}} + \frac{1}{\mid x + y - 2k \mid^{1+\alpha}} \right)$$

where c_α is a constant depending only on α. Then for all smooth functions $u, \varphi : \Omega \to \mathbb{R}$

$$\int_\Omega I(u)(x)\varphi(x)dx = \int_\Omega u(x)I(\varphi)(x)dx$$

4. Regularized problem

We consider the following regularized problem

$$\begin{cases} \partial_t u + \partial_x(f_\epsilon(u)\partial_x I(u)) = 0 & \text{for } x \in \Omega, \quad t > 0, \\ \partial_x u = 0, f_\epsilon(u)\partial_x I(u) = 0 & \text{for } x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x) & \text{for } x \in \Omega, \end{cases} \qquad (2)$$

where $f_\epsilon(s) = s_+^n + \epsilon, \quad \epsilon > 0$ and $0 < \alpha < 2$.

Stationnary problem

For $\tau > 0, g \in H^{\frac{\alpha}{2}}(\Omega)$, find $u \in H_N^{\alpha+1}(\Omega)$ s.t.

$$\begin{cases} u + \tau\partial_x(f_\epsilon(u)\partial_x I(u)) = g & \text{in } \Omega, \\ \partial_x u = 0, \partial_x I(u) = 0 & \text{on } \partial\Omega. \end{cases} \qquad (3)$$

Once we get a solution for (3), we can prove the existence of a solution for (2).
We prove the existence of $u^\tau \in L^\infty(0,T; H^{\frac{\alpha}{2}}(\Omega))$ such that for all $\varphi \in C_c^1(0,T; H^1(\Omega))$,

$$\iint_{Q_{\tau,T}} \frac{u^\tau - S_\tau u^\tau}{\tau}\varphi dxdt = \iint_{Q_{\tau,T}} f_\epsilon(u^\tau)\partial_x I(u^\tau)\partial_x\varphi dxdt$$

where $S_\tau u^\tau(t,x) = u^\tau(t-\tau, x)$ and $Q_{\tau,T} = (\tau, T) \times \Omega$.

Existence of solutions for the regularized problem

Let $\tau \to 0$. For $0 < \alpha \leq 1$ we have

- $u^\tau \to u^\epsilon$ in $L^2(0,T; H_N^s(\Omega))$ for all $s < 1 + \alpha$,
- $I(u^\tau) \to I(u^\epsilon)$ in $L^2(0,T; L^q(\Omega))$ for all $q < \infty$.

For $1 < \alpha < 2$ we have

- $u^\tau \to u^\epsilon$ locally uniformly,
- $\partial_x I(u^\tau) \rightharpoonup \partial_x I(u^\epsilon)$ in $L^2(Q)$-weakly.

So, there exists $u^\epsilon \in L^\infty(0,T; H^{\frac{\alpha}{2}}(\Omega)) \cap L^2(0,T; H_N^{\alpha+1}(\Omega))$ s.t.

$$\iint_Q u^\epsilon\partial_t\varphi dxdt + \iint_Q f_\epsilon(u^\epsilon)\partial_x I(u^\epsilon)\partial_x\varphi dxdt = -\int_\Omega u_0\varphi(0,.)dx$$

for all $\varphi \in C^1(0,T; H^1(\Omega))$ with support in $[0,T) \times \bar{\Omega}$.

5. Existence of nonnegative solutions

First step: Let $\epsilon \to 0$. For $0 < \alpha \leq 1$ we have

$$u^\epsilon \to u \text{ in } C^0(0,T; L^p(\Omega)) \text{ for all } p < \frac{2}{1-\alpha},$$
$$I(u^\epsilon) \to I(u) \text{ in } L^2(0,T; L^q(\Omega)) \text{ for all } q < \infty,$$
$$\partial_x u^\epsilon \to \partial_x u \text{ in } L^2(0,T; L^p(\Omega)) \text{ for all } p < \frac{2}{1-\alpha}.$$

Using these convergences we can pass to the limit in the nonlinear term and prove that

$$(u^\epsilon)_+^n\partial_x I(u^\epsilon) \rightharpoonup u_+^n\partial_x I(u) \text{ in } L^2(0,T; L^m(\Omega)) \text{ weakly.}$$

and we get the solution u as in Th.1.
For $1 < \alpha < 2$ and for u_0 s.t. $\int_\Omega G(u_0)dx \leq c$ we have

$$u^\epsilon \to u \text{ locally uniformly,}$$
$$I(u^\epsilon) \to I(u) \text{ in } L^2(0,T; L^2(\Omega)),$$
$$\partial_x u^\epsilon \to \partial_x u \text{ in } L^2(0,T; \text{locally uniformly with respect to x}).$$

Using these convergences we can pass to the limit as in the first case and get a solution u as in Th.1.
Now let us consider $u_0 \in H^{\frac{\alpha}{2}}(\Omega)$ without the additional condition. We define

$$u_{0\delta}(x) = u_0(x) + \delta \quad (\text{satisfies } \int_\Omega G(u_{0\delta})dx \leq c)$$

and denote u_δ the nonnegative solution u constructed in the first step for the initial data $u_{0\delta}$. We prove that

$$u_\delta \in C_{t,x}^{\frac{\alpha-1}{2(\alpha+2)},\frac{\alpha-1}{2}}(Q)$$

So taking a subsequence we have

$$u_\delta \to u \quad \text{locally uniformly in } Q,$$

Finally, we prove that the nonlinear term $h_\delta = u_\delta^n\partial_x I(u_\delta)$ weakly converges to h in $L^2(Q)$ and

$$h = \begin{cases} u^n\partial_x I(u) & \text{in } Q_+ := \{u > 0\} \cap Q \\ 0 & \text{elsewhere.} \end{cases}$$

Thus we get the solution u as in Th.2.
Second step: u is nonnegative.
Consider a solution u constructed as in Th.1. We have

$$\limsup_{\epsilon\to 0} \int_\Omega G_\epsilon(u^\epsilon(t,x))dx < \infty.$$

Using this information we prove by contradiction that for all $\delta > 0$ and all $\eta > 0$, we have

$$\mid \{u(t,.) \leq -2\delta\} \mid \leq \eta.$$

Hence the set

$$\{u(t,.) < 0\} = \bigcup_{k\geq 1} \left\{ u(t,.) \leq \frac{-1}{k} \right\}$$

has measure zero and so $u(t,x) \geq 0$ for almost every $x \in \Omega$ and for all $t > 0$.

References

[1] Imbert, Cyril and Mellet, Antoine Existence of solutions for a higher order non-local equation appearing in crack dynamics. *Nonlinearity*, 12:3487–3514, 2011.

[2] Tarhini, Rana Study of a family of higher order non-local degenerate parabolic equations: from the porous medium equation to the thin film equation. *Journal of Differential Equations*, 259:5782–5812, 2015.

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