Heat kernels of non-symmetric jump processes: beyond the stable case

Renming Song

University of Illinois

3rd Conference on Nonlocal Operators
and Partial Differential Equations
June 27–July 1, 2016

Table of Contents

Table of Contents

Joint work with P. Kim and Z. Vondraček: arXiv:1606.02005

Suppose $d \geq 1$, $\alpha \in (0,2)$ and κ is a Borel function on $\mathbb{R}^d \times \mathbb{R}^d$ such that

$$0 < \kappa_0 \leq \kappa(x,z) \leq \kappa_1 \,, \quad \kappa(x,z) = \kappa(x,-z) \,, \tag{1}$$

and for some $\beta \in (0,1)$,

$$|\kappa(x,z) - \kappa(y,z)| \leq \kappa_2 |x-y|^\beta \,. \tag{2}$$

Define

$$\mathcal{L}_\alpha^\kappa f(x) = \lim_{\varepsilon \downarrow 0} \int_{\{z \in \mathbb{R}^d : |z| > \varepsilon\}} (f(x+z) - f(z)) \frac{\kappa(x,z)}{|z|^{d+\alpha}} \, dz \,. \tag{3}$$

This is a non-symmetric and non-local stable-like operator.

These operator can be regarded as the non-local counterpart of elliptic operators in non-divergence form. In this context the Hölder continuity of $\kappa(\cdot, z)$ in (2) is a natural assumption.

These operator can be regarded as the non-local counterpart of elliptic operators in non-divergence form. In this context the Hölder continuity of $\kappa(\cdot, z)$ in (2) is a natural assumption.

In [PTRF16], Z.-Q. Chen and X. Zhang proved the existence and uniqueness of a non-negative jointly continuous function $p_\alpha^\kappa(t,x,y)$ in $(t,x,y) \in (0,1] \times \mathbb{R}^d \times \mathbb{R}^d$ solving the equation

$$\partial_t p_\alpha^\kappa(t,x,y) = \mathcal{L}_\alpha^\kappa p_\alpha^\kappa(t,\cdot,y)(x), \quad x \neq y,$$

and satisfying four properties - an upper bound, Hölder's estimate, fractional derivative estimate and continuity.

These operator can be regarded as the non-local counterpart of elliptic operators in non-divergence form. In this context the Hölder continuity of $\kappa(\cdot, z)$ in (2) is a natural assumption.

In [PTRF16], Z.-Q. Chen and X. Zhang proved the existence and uniqueness of a non-negative jointly continuous function $p^\kappa_\alpha(t,x,y)$ in $(t,x,y) \in (0,1] \times \mathbb{R}^d \times \mathbb{R}^d$ solving the equation

$$\partial_t p^\kappa_\alpha(t,x,y) = \mathcal{L}^\kappa_\alpha p^\kappa_\alpha(t,\cdot,y)(x)\,, \quad x \neq y\,,$$

and satisfying four properties - an upper bound, Hölder's estimate, fractional derivative estimate and continuity.

Their main result is as follows:

Theorem (Chen-Zhang)

There exists a unique non-negative jointly continuous function $p_\alpha^\kappa(t,x,y)$, $(t,x,y) \in (0,1] \times \mathbb{R}^d \times \mathbb{R}^d$, solving

$$\partial_t p_\alpha^\kappa(t,x,y) = \mathcal{L}_\alpha^\kappa p_\alpha^\kappa(t,\cdot,y)(x)\,, \quad x \neq y\,, \tag{4}$$

and satisfying the following properties:

Theorem (Chen-Zhang)

There exists a unique non-negative jointly continuous function $p^{\kappa}_{\alpha}(t,x,y)$, $(t,x,y)\in(0,1]\times\mathbb{R}^d\times\mathbb{R}^d$, solving

$$\partial_t p^{\kappa}_{\alpha}(t,x,y)=\mathcal{L}^{\kappa}_{\alpha}p^{\kappa}_{\alpha}(t,\cdot,y)(x)\,,\quad x\neq y\,, \tag{4}$$

and satisfying the following properties:

(i) There exists $c_1>0$ such that for all $t\in(0,1]$ and $x,y\in\mathbb{R}^d$,

$$p^{\kappa}_{\alpha}(t,x,y)\leq c_1 t(t^{\frac{1}{\alpha}}+|x-y|)^{-d-\alpha}.$$

Theorem (Chen-Zhang)

There exists a unique non-negative jointly continuous function $p^\kappa_\alpha(t,x,y)$, $(t,x,y)\in(0,1]\times\mathbb{R}^d\times\mathbb{R}^d$, solving

$$\partial_t p^\kappa_\alpha(t,x,y)=\mathcal{L}^\kappa_\alpha p^\kappa_\alpha(t,\cdot,y)(x)\,,\quad x\neq y\,, \tag{4}$$

and satisfying the following properties:

(i) There exists $c_1>0$ such that for all $t\in(0,1]$ and $x,y\in\mathbb{R}^d$,

$$p^\kappa_\alpha(t,x,y)\le c_1 t(t^{\frac{1}{\alpha}}+|x-y|)^{-d-\alpha}.$$

(ii) For every $\gamma\in(0,\alpha\wedge 1)$, there exists $c_2>0$ such that for all $t\in(0,1]$ and $x,x',y\in\mathbb{R}^d$,

$$|p^\kappa_\alpha(t,x,y)-p^\kappa_\alpha(t,x',y)|\le c_2|x-x'|^\gamma t^{1-\frac{\gamma}{\alpha}}(t^{\frac{1}{\alpha}}+|x-y|\wedge|x'-y|)^{-d-\alpha}.$$

Theorem (Chen-Zhang) (cont)

(iii) For all $x \neq y$ in $\mathbb{R}^d$, the map $t \mapsto \mathcal{L}_\alpha^\kappa p_\alpha^\kappa(t, \cdot, y)(x)$ is continuous in $(0, 1]$ and

$$|\mathcal{L}_\alpha^\kappa p_\alpha^\kappa(t, \cdot, y)(x)| \leq c_3(t^{\frac{1}{\alpha}} + |x - y|)^{-d-\alpha}.$$

Theorem (Chen-Zhang) (cont)

(iii) For all $x \neq y$ in $\mathbb{R}^d$, the map $t \mapsto \mathcal{L}^\kappa_\alpha p^\kappa_\alpha(t, \cdot, y)(x)$ is continuous in $(0, 1]$ and

$$|\mathcal{L}^\kappa_\alpha p^\kappa_\alpha(t, \cdot, y)(x)| \leq c_3(t^{\frac{1}{\alpha}} + |x - y|)^{-d-\alpha}.$$

(iv) For any bounded and uniformly continuous function $f : \mathbb{R}^d \to \mathbb{R}$,

$$\lim_{t \downarrow 0} \sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} p^\kappa_\alpha(t, x, y) f(y)\, dy - f(x) \right| = 0\,.$$

Theorem (Chen-Zhang) (cont)

(iii) For all $x \neq y$ in $\mathbb{R}^d$, the map $t \mapsto \mathcal{L}_\alpha^\kappa p_\alpha^\kappa(t,\cdot,y)(x)$ is continuous in $(0,1]$ and

$$|\mathcal{L}_\alpha^\kappa p_\alpha^\kappa(t,\cdot,y)(x)| \leq c_3(t^{\frac{1}{\alpha}} + |x-y|)^{-d-\alpha}.$$

(iv) For any bounded and uniformly continuous function $f : \mathbb{R}^d \to \mathbb{R}$,

$$\lim_{t \downarrow 0} \sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} p_\alpha^\kappa(t,x,y) f(y)\, dy - f(x) \right| = 0\,.$$

Moreover, the following conclusions are valid:

Theorem (Chen-Zhang) (cont)

(1) For all $(t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} p_\alpha^\kappa(t, x, y) dy = 1.$$

Theorem (Chen-Zhang) (cont)

(1) For all $(t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} p_\alpha^\kappa(t, x, y) dy = 1.$$

(2) For all $s, t \in (0, 1]$ with $s + t \in (0, 1]$, and all $x, y \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} p_\alpha^\kappa(s, x, z) p_\alpha^\kappa(t, x, y) dz = p_\alpha^\kappa(s + t, x, y).$$

Theorem (Chen-Zhang) (cont)

(1) For all $(t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} p^\kappa_\alpha(t, x, y)dy = 1.$$

(2) For all $s, t \in (0, 1]$ with $s + t \in (0, 1]$, and all $x, y \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} p^\kappa_\alpha(s, x, z)p^\kappa_\alpha(t, x, y)dz = p^\kappa_\alpha(s + t, x, y).$$

(3) There exists $c_4 > 0$ such that for all $t \in (0, 1]$ and $x, y \in \mathbb{R}^d$,

$$p^\kappa_\alpha(t, x, y) \geq c_4 t(t^{\frac{1}{\alpha}} + |x - y|)^{-d-\alpha}.$$

Theorem (Chen-Zhang) (cont)

(4) For any $f \in C_b^2(\mathbb{R}^d)$,

$$\lim_{t \downarrow 0} \frac{1}{t} \int_{\mathbb{R}^d} p_\alpha^\kappa(t,x,y)(f(y)-f(x))dy = \mathcal{L}_\alpha^\kappa f(x)$$

and the convergence is uniform.

Theorem (Chen-Zhang) (cont)

(4) For any $f \in C_b^2(\mathbb{R}^d)$,

$$\lim_{t\downarrow 0} \frac{1}{t} \int_{\mathbb{R}^d} p_\alpha^\kappa(t,x,y)(f(y)-f(x))dy = \mathcal{L}_\alpha^\kappa f(x)$$

and the convergence is uniform.

(5) The C_0-semigroup $(P_t^\kappa : t \geq 0)$ defined by

$$P_t^\kappa f(x) = \int_{\mathbb{R}^d} p_\alpha^\kappa(t,x,y) f(y) dy$$

is analytic in $L^p(\mathbb{R}^d)$ for all $p \in [1,\infty)$.

In Professor Knopova's talk yesterday, she reported her joint work with Bogdan and Sztonyk on Lévy-type processes associated with a stable dominated Lévy-type kernel.

In Professor Knopova's talk yesterday, she reported her joint work with Bogdan and Sztonyk on Lévy-type processes associated with a stable dominated Lévy-type kernel.

In another direction, most of results on isotropic α-stable process have been generalized to more general symmetric Lévy processes, like subordinate Brownian motions, or unimodal Lévy processes, like reported in Professor Ryznar's talk.

In Professor Knopova's talk yesterday, she reported her joint work with Bogdan and Sztonyk on Lévy-type processes associated with a stable dominated Lévy-type kernel.

In another direction, most of results on isotropic α-stable process have been generalized to more general symmetric Lévy processes, like subordinate Brownian motions, or unimodal Lévy processes, like reported in Professor Ryznar's talk.

Our goal is to extend the results of the Chen-Zhang paper to more general operators than the ones defined in (3). These operators will be non-symmetric and not necessarily stable-like. We will replace the kernel $\kappa(x,z)|z|^{-d-\alpha}$ with a kernel $\kappa(x,z)J(z)$ where κ still satisfies (1) and (2), but $J(z)$ is the Lévy density of a rather general symmetric Lévy process.

Setting and assumptions

Suppose that $S = (S_t : t \geq 0)$ is a subordinator, that is, a non-negative Lévy process with $S_0 = 0$. Let ϕ be the Laplace exponent of S, that is,

$$\mathbb{E}e^{-\lambda S_t} = e^{-t\phi(\lambda)}, \quad t > 0, \lambda > 0.$$

Setting and assumptions

Suppose that $S = (S_t : t \geq 0)$ is a subordinator, that is, a non-negative Lévy process with $S_0 = 0$. Let ϕ be the Laplace exponent of S, that is,

$$\mathbb{E}e^{-\lambda S_t} = e^{-t\phi(\lambda)}, \quad t > 0, \lambda > 0.$$

A function $\phi : (0, \infty) \mapsto (0, \infty)$ is the Laplace exponent of a subordinator iff it is a Bernstein function satisfying $\lim_{\lambda \to 0} \phi(\lambda) = 0$. Recall that a nonnegative function ϕ on $(0, \infty)$ is a Bernstein function if it is C^∞ and $(-1)^{n-1}\phi^{(n)} \geq 0$.

Setting and assumptions II

Suppose that S has no drift. Then ϕ admits the following expression

$$\phi(\lambda) = \int_0^\infty (1 - e^{-\lambda t})\mu(dt), \quad \lambda > 0,$$

where μ is a measure on $(0, \infty)$ such that $\int_0^\infty (1 \wedge t)\mu(dt) < \infty$. μ is called the Lévy measure of S.

Setting and assumptions II

Suppose that S has no drift. Then ϕ admits the following expression

$$\phi(\lambda) = \int_0^\infty (1 - e^{-\lambda t})\mu(dt), \quad \lambda > 0,$$

where μ is a measure on $(0, \infty)$ such that $\int_0^\infty (1 \wedge t)\mu(dt) < \infty$. μ is called the Lévy measure of S.

Suppose that $B = (B_t : t \geq 0)$ is a Brownian motion in R^d with generator Δ. Suppose that B and S are independent. Then the process $(X_t : t \geq 0)$ defined by $X_t := B_{S_t}$ is a Lévy process and it is called a subordinate Brownian motion. The generator X is $-\phi(-\Delta)$.

Setting and assumptions III

When $\phi(\lambda) = \lambda^{\alpha/2}$, the resulting subordinate Brownian motion is a symmetric α-stable process. Without loss of generality, we will assume $\phi(1) = 1$.

Setting and assumptions III

When $\phi(\lambda) = \lambda^{\alpha/2}$, the resulting subordinate Brownian motion is a symmetric α-stable process. Without loss of generality, we will assume $\phi(1) = 1$.

The Lévy exponent of X is $\Phi(\xi) = \phi(|\xi|^2)$. The Lévy measure of X has a density $J(x) = J(|x|)$ with

$$J(r) = \int_{(0,\infty)} (4\pi t)^{-d/2} e^{-\frac{r^2}{4t}} \mu(dt)\,, \quad r > 0.$$

Setting and assumptions IV

Our main assumption is the following *weak lower scaling condition at infinity*: There exist $\delta_1 \in (0, 2]$ and $a_1 \in (0, 1)$ such that

$$a_1 \lambda^{\delta_1} \Phi(r) \leq \Phi(\lambda r) , \quad \lambda \geq 1, r \geq 1 . \tag{5}$$

This condition implies that $\lim_{\lambda \to \infty} \Phi(\lambda) = \infty$ and hence $\int_{\mathbb{R}^d \setminus \{0\}} j(|y|) dy = \infty$.

Setting and assumptions IV

Our main assumption is the following *weak lower scaling condition at infinity*: There exist $\delta_1 \in (0, 2]$ and $a_1 \in (0, 1)$ such that

$$a_1 \lambda^{\delta_1} \Phi(r) \le \Phi(\lambda r), \quad \lambda \ge 1, r \ge 1. \tag{5}$$

This condition implies that $\lim_{\lambda \to \infty} \Phi(\lambda) = \infty$ and hence $\int_{\mathbb{R}^d \setminus \{0\}} j(|y|)dy = \infty$.

The weak lower scaling condition governs the short-time small-space behavior of the subordinate Brownian motion. We also need a weak condition on the behavior of Φ near zero. We assume that

$$\int_0^1 \frac{\Phi(r)}{r} \, dr = C_* < \infty. \tag{6}$$

Setting and assumptions V

Assume that κ is a function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying (1) and (2). We define

$$\mathcal{L}^{\kappa} f(x) := \lim_{\varepsilon \downarrow 0} \mathcal{L}^{\kappa,\varepsilon} f(x)$$

where

$$\mathcal{L}^{\kappa,\varepsilon} f(x) := \int_{|z|>\varepsilon} (f(x+z) - f(z))\kappa(x,z)J(z)\, dz, \quad \varepsilon > 0.$$

Setting and assumptions V

Assume that κ is a function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying (1) and (2). We define

$$\mathcal{L}^\kappa f(x) := \lim_{\varepsilon \downarrow 0} \mathcal{L}^{\kappa,\varepsilon} f(x)$$

where

$$\mathcal{L}^{\kappa,\varepsilon} f(x) := \int_{|z|>\varepsilon} (f(x+z) - f(z))\kappa(x,z)J(z)\, dz, \quad \varepsilon > 0.$$

For $t > 0$ and $\in \mathbb{R}^d$ we define

$$\rho(t,x) = \rho^{(d)}(t,x) := \Phi\left(\left(\frac{1}{\Phi^{-1}(t^{-1})} + |x|\right)^{-1}\right)\left(\frac{1}{\Phi^{-1}(t^{-1})} + |x|\right)^{-d}.$$

In case when $\Phi(r) = r^\alpha$ we see that $\rho(t,x) = (t^{1/\alpha} + |x|)^{-d-\alpha}$.

Theorem 1

There exists a unique non-negative jointly continuous function $p^\kappa(t,x,y)$ on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$ solving

$$\partial_t p^\kappa(t,x,y) = \mathcal{L}^\kappa p^\kappa(t,\cdot,y)(x)\,, \quad x\neq y\,,$$

and satisfying the following properties:

Theorem 1

There exists a unique non-negative jointly continuous function $p^\kappa(t,x,y)$ on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$ solving

$$\partial_t p^\kappa(t,x,y)=\mathcal{L}^\kappa p^\kappa(t,\cdot,y)(x)\,,\quad x\neq y\,,$$

and satisfying the following properties:

(i) For every $T\geq 1$, there is a constant $c_1>0$ so that for all $t\in(0,T]$ and $x,y\in\mathbb{R}^d$,

$$p^\kappa(t,x,y)\leq c_1 t\rho(t,x-y)\,.$$

Theorem 1 (cont)

(ii) For any $x, y \in \mathbb{R}^d$, $x \neq y$, the mapping $t \mapsto \mathcal{L}^\kappa p^\kappa(t, \cdot, y)(x)$ is continuous in $(0, \infty)$, and, for each $T > 0$ there is a constant $c_2 > 0$ so that for all $t \in (0, T]$, $\varepsilon \in [0, 1]$ and $x, y \in \mathbb{R}^d$,

$$|\mathcal{L}^{\kappa,\varepsilon} p^\kappa(t, \cdot, y)(x)| \leq c_2 \rho(t, x - y) \,.$$

Theorem 1 (cont)

(ii) For any $x, y \in \mathbb{R}^d$, $x \neq y$, the mapping $t \mapsto \mathcal{L}^\kappa p^\kappa(t, \cdot, y)(x)$ is continuous in $(0, \infty)$, and, for each $T > 0$ there is a constant $c_2 > 0$ so that for all $t \in (0, T]$, $\varepsilon \in [0, 1]$ and $x, y \in \mathbb{R}^d$,

$$|\mathcal{L}^{\kappa,\varepsilon} p^\kappa(t, \cdot, y)(x)| \leq c_2 \rho(t, x - y) .$$

(iii) For any bounded and uniformly continuous function $f : \mathbb{R}^d \to \mathbb{R}$,

$$\lim_{t \downarrow 0} \sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} p^\kappa(t, x, y) f(y) \, dy - f(x) \right| = 0 .$$

Theorem 2

The following conclusions are also valid:

Theorem 2

The following conclusions are also valid:

(1) For all $(t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$, $p^\kappa(t, x, y) \geq 0$ and

$$\int_{\mathbb{R}^d} p^\kappa(t, x, y)\, dy = 1 .$$

Theorem 2

The following conclusions are also valid:

(1) For all $(t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$, $p^\kappa(t, x, y) \geq 0$ and

$$\int_{\mathbb{R}^d} p^\kappa(t, x, y)\, dy = 1\,.$$

(2) For all $s, t > 0$ and all $x, y \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} p^\kappa(t, x, z) p^\kappa(s, z, y)\, dz = p^\kappa(t + s, x, y)\,.$$

Theorem 2

(3) For every $T \geq 1$, there is a constant $c_4 > 0$ such that for all $0 < s \leq t \leq T$ and $x, x', y \in \mathbb{R}^d$ with either $x \neq y$ or $x' \neq y$,

$$
\begin{aligned}
&|p^\kappa(s, x, y) - p^\kappa(t, x', y)| \\
&\leq c_4 \left(|t - s| + |x - x'| t\Phi^{-1}(t^{-1})\right) \left(\rho(s, x - y) \vee \rho(s, x' - y)\right).
\end{aligned}
$$

Theorem 2

(3) For every $T \geq 1$, there is a constant $c_4 > 0$ such that for all $0 < s \leq t \leq T$ and $x, x', y \in \mathbb{R}^d$ with either $x \neq y$ or $x' \neq y$,

$$\begin{aligned} & |p^\kappa(s,x,y) - p^\kappa(t,x',y)| \\ & \leq c_4 \left(|t-s| + |x-x'| t\Phi^{-1}(t^{-1})\right) \left(\rho(s,x-y) \vee \rho(s,x'-y)\right). \end{aligned}$$

(4) For every $T \geq 1$, there exists $c_5 => 0$ so that for all $x, y \in \mathbb{R}^d$, $x \neq y$, and $t \in (0, T]$,

$$|\nabla_x p^\kappa(t,x,y)| \leq c_5 \Phi^{-1}(t^{-1}) t \rho(t, x-y).$$

Theorem 3

(a) Let $\varepsilon > 0$. For any $f \in C_b^{2,\varepsilon}(\mathbb{R}^d)$, we have

$$\lim_{t \downarrow 0} \frac{1}{t} \left(P_t^\kappa f(x) - f(x) \right) = \mathcal{L}^\kappa f(x) \,,$$

and the convergence is uniform.

Theorem 3

(a) Let $\varepsilon > 0$. For any $f \in C_b^{2,\varepsilon}(\mathbb{R}^d)$, we have

$$\lim_{t \downarrow 0} \frac{1}{t} \left(P_t^\kappa f(x) - f(x) \right) = \mathcal{L}^\kappa f(x) ,$$

and the convergence is uniform.

(b) The semigroup $(P_t^\kappa)_{t \geq 0}$ of $\mathcal{L}^\kappa$ is analytic in $L^p(\mathbb{R}^d)$ for every $p \in [1, \infty)$.

For a lower bound, we will assume the following weak upper scaling condition: there exist $\delta_2 \in (0,2)$ and $a_2 > 0$ such that

$$\Phi(\lambda r) \le a_2 \lambda^{\delta_2} \Phi(r)\,, \quad \lambda \ge 1, r \ge 1\,. \tag{7}$$

Theorem 4

For a lower bound, we will assume the following weak upper scaling condition: there exist $\delta_2 \in (0,2)$ and $a_2 > 0$ such that

$$\Phi(\lambda r) \le a_2 \lambda^{\delta_2} \Phi(r) , \quad \lambda \ge 1, r \ge 1 . \tag{7}$$

Theorem 4

For every $T \ge 1$, there exists $c_6 > 0$ such that for all $t \in (0, T]$

$$p^{\kappa}(t,x,y) \ge c_6 \begin{cases} \Phi^{-1}(t^{-1})^d & \text{if } |x-y| \le 3\Phi^{-1}(t^{-1})^{-1}, \\ tj\left(|x-y|\right) & \text{if } |x-y| > 3\Phi^{-1}(t^{-1})^{-1}. \end{cases}$$

In particular, for every $T, M \ge 1$, there exists $c_7 > 0$ for all $t \in (0, T]$ and $x, y \in \mathbb{R}^d$ with $|x-y| \le M$,

$$p^{\kappa}(t,x,y) \ge c_7 t \rho(t, x-y) .$$

When the global lower and upper scaling conditions are both satisfied, the lower and upper bound differ by a multiplicative constant.

When the global lower and upper scaling conditions are both satisfied, the lower and upper bound differ by a multiplicative constant.

Here are some examples:
(i) $\phi(\lambda) = \lambda^{\alpha_1} + \lambda^{\alpha_2}$, $0 < \alpha_1 < \alpha_2 < 1$;
(ii) $\phi(\lambda) = (\lambda + \lambda^{\alpha_1})^{\alpha_2}$, $\alpha_1, \alpha_2 \in (0, 1)$;
(iii) $\phi(\lambda) = (\lambda + m^{1/\alpha})^{\alpha} - m$, $\alpha \in (0, 1)$, $m > 0$;
(iv) $\phi(\lambda) = \lambda^{\alpha_1}(\log(1 + \lambda))^{\alpha_2}$, $\alpha_1 \in (0, 1)$, $\alpha_2 \in (0, 1 - \alpha_1]$;
(v) $\phi(\lambda) = \lambda^{\alpha_1}(\log(1 + \lambda))^{-\alpha_2}$, $\alpha_1 \in (0, 1)$, $\alpha_2 \in (0, \alpha_1)$;
(vi) $\phi(\lambda) = \lambda / \log(1 + \lambda^{\alpha})$, $\alpha \in (0, 1)$.

When the global lower and upper scaling conditions are both satisfied, the lower and upper bound differ by a multiplicative constant.

Here are some examples:
(i) $\phi(\lambda) = \lambda^{\alpha_1} + \lambda^{\alpha_2}$, $0 < \alpha_1 < \alpha_2 < 1$;
(ii) $\phi(\lambda) = (\lambda + \lambda^{\alpha_1})^{\alpha_2}$, $\alpha_1, \alpha_2 \in (0, 1)$;
(iii) $\phi(\lambda) = (\lambda + m^{1/\alpha})^{\alpha} - m$, $\alpha \in (0, 1)$, $m > 0$;
(iv) $\phi(\lambda) = \lambda^{\alpha_1}(\log(1 + \lambda))^{\alpha_2}$, $\alpha_1 \in (0, 1)$, $\alpha_2 \in (0, 1 - \alpha_1]$;
(v) $\phi(\lambda) = \lambda^{\alpha_1}(\log(1 + \lambda))^{-\alpha_2}$, $\alpha_1 \in (0, 1)$, $\alpha_2 \in (0, \alpha_1)$;
(vi) $\phi(\lambda) = \lambda / \log(1 + \lambda^{\alpha})$, $\alpha \in (0, 1)$.

The functions in (i)–(v) satisfy (5), (6) and (7); while the function in (vi) satisfies (5) and (6), but does not satisfy (7). The function $\phi(\lambda) = \lambda / \log(1 + \lambda)$ satisfies (5), but does not satisfy the other two conditions.

We follow the ideas and the road-map from the Chen-Zhang paper, and use the freezing coefficient method. At many stages we encounter substantial technical difficulties due to the fact that in the stable-like case one deals with power functions while in the present situation the power functions are replaced with a quite general Φ and its variants.

We follow the ideas and the road-map from the Chen-Zhang paper, and use the freezing coefficient method. At many stages we encounter substantial technical difficulties due to the fact that in the stable-like case one deals with power functions while in the present situation the power functions are replaced with a quite general Φ and its variants.

Here is a lemma that will give a flavor of the things we have to deal with. For $\gamma, \beta \in \mathbb{R}$, let

$$\rho_\gamma^\beta(t,x) := \Phi^{-1}(t^{-1})^{-\gamma}(|x|^\beta \wedge 1)\rho(t,x)\,, \quad t > 0, x \in \mathbb{R}^d\,.$$

Note that $\rho_0^0(t,x) = \rho(t,x)$.

Lemma

(a) For every $T \geq 1$, there exists $c > 0$ such that for $0 < t \leq 1$, all $\beta \in [0, \delta_1/2]$ and $\gamma \in \mathbb{R}$,

$$\int_{\mathbb{R}^d} \rho_\gamma^\beta(t,x)\, dx \leq c t^{-1} \Phi^{-1}(t^{-1})^{-\gamma-\beta}\,.$$

Lemma

(a) For every $T \geq 1$, there exists $c > 0$ such that for $0 < t \leq 1$, all $\beta \in [0, \delta_1/2]$ and $\gamma \in \mathbb{R}$,

$$\int_{\mathbb{R}^d} \rho_\gamma^\beta(t,x)\, dx \leq c t^{-1} \Phi^{-1}(t^{-1})^{-\gamma-\beta}\,.$$

(b) For every $T \geq 1$, there exists $C_0 > 0$ such that for all $\beta_1, \beta_2 \geq 0$ with $\beta_1 + \beta_2 \leq \delta_1/2$, $\gamma_1, \gamma_2 \in \mathbb{R}$ and $0 < s < t \leq 1$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} \rho_{\gamma_1}^{\beta_1}(t-s, x-z)\rho_{\gamma_2}^{\beta_2}(s,z)\, dz \\
&\leq C_0\Big((t-s)^{-1}\Phi^{-1}((t-s)^{-1})^{-\gamma_1-\beta_1-\beta_2}\Phi^{-1}(s^{-1})^{-\gamma_2} \\
&\quad + \Phi^{-1}((t-s)^{-1})^{-\gamma_1} s^{-1}\Phi^{-1}(s^{-1})^{-\gamma_2-\beta_1-\beta_2}\Big)\rho(t,x) \\
&+ C_0(t-s)^{-1}\Phi^{-1}((t-s)^{-1})^{-\gamma_1-\beta_1}\Phi^{-1}(s^{-1})^{-\gamma_2}\rho_0^{\beta_2}(t,x) \\
&+ C_0\Phi^{-1}((t-s)^{-1})^{-\gamma_1} s^{-1}\Phi^{-1}(s^{-1})^{-\gamma_2-\beta_2}\rho_0^{\beta_1}(t,x)\,.
\end{aligned}$$

Lemma (cont)

(c) For all $\beta_1, \beta_2 \geq 0$ with $\beta_1 + \beta_2 \leq \delta_1/2$, $\theta, \eta \in [0, 1]$, $\gamma_1 + \beta_1 + 2 - 2\theta > 0$, $\gamma_2 + \beta_2 + 2 - 2\eta > 0$, $0 < t \leq T$ and $x \in \mathbb{R}^d$, we have

$$\int_0^t \int_{\mathbb{R}^d} (t-s)^{1-\theta} \rho_{\gamma_1}^{\beta_1}(t-s, x-z) s^{1-\eta} \rho_{\gamma_2}^{\beta_2}(s, z)\, dz\, ds$$
$$\leq c_2 t^{2-\theta-\eta} \left(\rho^0_{\gamma_1+\gamma_2+\beta_1+\beta_2} + \rho^{\beta_1}_{\gamma_1+\gamma_2+\beta_2} + \rho^{\beta_2}_{\gamma_1+\gamma_2+\beta_1} \right)(t, x).$$

where

$$c_2 = 4C_0(T) B\left((\gamma_1 + \beta_1)/2 + 1 - \theta, \gamma_2 + \beta_2/2 + 1 - \eta\right).$$

Let $\mathfrak{K} : \mathbb{R}^d \to (0, \infty)$ be a symmetric function, that is, $\mathfrak{K}(z) = \mathfrak{K}(-z)$. Assume that there are $0 < \kappa_0 \le \kappa_1 < \infty$ such that

$$\kappa_0 \le \mathfrak{K}(z) \le \kappa_1 , \qquad \text{for all } z \in \mathbb{R}^d . \tag{8}$$

Let $j^{\mathfrak{K}}(z) := \mathfrak{K}(z)J(z)$, $z \in \mathbb{R}^d$. Let $Z^{\mathfrak{K}}$ be the purely discontinuous Lévy process with Lévy measure $j^{\mathfrak{K}}(z)$. The infinitesimal generator of $Z^{\mathfrak{K}}$ is given by

$$\mathcal{L}^{\mathfrak{K}} f(x) = \text{p.v.} \int_{\mathbb{R}^d} (f(x+z) - f(x)) \mathfrak{K}(z) J(z) \, dz.$$

Let $\mathfrak{K} : \mathbb{R}^d \to (0, \infty)$ be a symmetric function, that is, $\mathfrak{K}(z) = \mathfrak{K}(-z)$. Assume that there are $0 < \kappa_0 \le \kappa_1 < \infty$ such that

$$\kappa_0 \le \mathfrak{K}(z) \le \kappa_1 \,, \qquad \text{for all } z \in \mathbb{R}^d \,. \tag{8}$$

Let $j^{\mathfrak{K}}(z) := \mathfrak{K}(z)J(z)$, $z \in \mathbb{R}^d$. Let $Z^{\mathfrak{K}}$ be the purely discontinuous Lévy process with Lévy measure $j^{\mathfrak{K}}(z)$. The infinitesimal generator of $Z^{\mathfrak{K}}$ is given by

$$\mathcal{L}^{\mathfrak{K}} f(x) = \mathrm{p.v.} \int_{\mathbb{R}^d} (f(x+z) - f(x))\mathfrak{K}(z)J(z)\, dz.$$

We first study the heat kernel estimates of this process.

For a fixed $y \in \mathbb{R}^d$, let $\mathfrak{K}_y(z) = \kappa(y, z)$ and let $\mathcal{L}^{\mathfrak{K}_y}$ be the freezing operator

$$\mathcal{L}^{\mathfrak{K}_y} f(x) = \mathcal{L}^{\mathfrak{K}_y,0} f(x) = \lim_{\varepsilon \downarrow 0} \mathcal{L}^{\mathfrak{K}_y,\varepsilon} f(x),$$

where

$$\mathcal{L}^{\mathfrak{K}_y,\varepsilon} f(x) = \int_{|z|>\varepsilon} \delta_f(x; z)\kappa(y, z)J(z)dz.$$

Let $p_y(t, x) = p^{\mathfrak{K}_y}(t, x)$ be the heat kernel of the operator $\mathcal{L}^{\mathfrak{K}_y}$.

For a fixed $y \in \mathbb{R}^d$, let $\mathfrak{K}_y(z) = \kappa(y,z)$ and let $\mathcal{L}^{\mathfrak{K}_y}$ be the freezing operator

$$\mathcal{L}^{\mathfrak{K}_y} f(x) = \mathcal{L}^{\mathfrak{K}_y,0} f(x) = \lim_{\varepsilon \downarrow 0} \mathcal{L}^{\mathfrak{K}_y,\varepsilon} f(x),$$

where

$$\mathcal{L}^{\mathfrak{K}_y,\varepsilon} f(x) = \int_{|z|>\varepsilon} \delta_f(x;z)\kappa(y,z)J(z)dz.$$

Let $p_y(t,x) = p^{\mathfrak{K}_y}(t,x)$ be the heat kernel of the operator $\mathcal{L}^{\mathfrak{K}_y}$.

Define

$$q_0(t,x,y) := \left(\mathcal{L}^{\mathfrak{K}_x} - \mathcal{L}^{\mathfrak{K}_y}\right) p_y(t,\cdot)(x-y)\,.$$

Then we solve the integral equation

$$q(t,x,y) = q_0(t,x,y) + \int_0^t \int_{\mathbb{R}^d} q_0(t-s,x,z)q(s,z,y)\,dz\,ds$$

by iteration:

$$q_n(t,x,y) := \int_0^t \int_{\mathbb{R}^d} q_0(t-s,x,z)q_{n-1}(s,z,y)\,dz\,ds$$

and

$$q(t,x,y) = \sum_{n=0}^{\infty} q_n(t,x,y).$$

Then we solve the integral equation

$$q(t,x,y) = q_0(t,x,y) + \int_0^t \int_{\mathbb{R}^d} q_0(t-s,x,z)q(s,z,y)\,dz\,ds$$

by iteration:

$$q_n(t,x,y) := \int_0^t \int_{\mathbb{R}^d} q_0(t-s,x,z)q_{n-1}(s,z,y)\,dz\,ds$$

and

$$q(t,x,y) = \sum_{n=0}^{\infty} q_n(t,x,y).$$

Finally, we define

$$p^\kappa(t,x,y) := p_y(t,x-y) + \int_0^t \int_{\mathbb{R}^d} p_z(t-s,x-z)q(s,z,y)\,dz\,ds\,.$$