Decay and Harnack estimates for fully non-local diffusion equations

Rico Zacher

Ulm University, Germany

partially joint work with D. Dier, J. Kemppainen, J. Siljander, V. Vergara

Bedlewo, June 2016

Non-local in time diffusion equations

Let $\Omega \subset \mathbb{R}^d$ be open. Consider

$$(1) \qquad \partial_t\Big(k * [u - u_0]\Big) - Lu = f \quad \text{in} \quad (0,\infty) \times \Omega,$$
$$u|_{t=0} = u_0 \quad \text{in} \quad \Omega.$$

Here L is an elliptic operator w.r.t. space, e.g. $L = -(-\Delta)^{\beta/2}$, $\beta \in (0,2]$, and $(k * v)(t) = \int_0^t k(t-\tau)v(\tau)\,d\tau$, $t > 0$. We assume

$(\mathcal{PC})$ $k \in L_{1,loc}(\mathbb{R}_+)$ is ≥ 0, $\searrow$, and $\exists l \in L_{1,loc}(\mathbb{R}_+)$ such that $l \geq 0$ and $k * l = 1$ in $(0,\infty)$. Write $(k,l) \in \mathcal{PC}$. ($\Rightarrow k(0+) = \infty$)

Example: $(k,l) = (g_{1-\alpha}, g_\alpha)$, $\alpha \in (0,1)$, where $g_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$, $t > 0$, $\beta > 0$. $\hookrightarrow$ $\partial_t(k * u) = \partial_t^\alpha u$ (Riemann-Liouville derivative of order α)

Note that $\partial_t(g_{1-\alpha} * [u - u(0)]) = g_{1-\alpha} * \dot{u}$ (Caputo fractional derivative), if u is suff. smooth.

Applications

• anomalous diffusion processes: Let $\Omega = \mathbb{R}^d$ and $L = \Delta$. $u(t,\cdot)$ describes the evolution of a pdf. Let $Z(t,x)$ be the fundamental solution of (1) with $Z(0,x) = \delta(x)$. If $(k,l) \in \mathcal{PC}$, $Z(t,\cdot)$ is a pdf for $t > 0$. Define

$$msd(t) = \int_{\mathbb{R}^d} |x|^2 Z(t,x)\,dx, \quad t > 0. \quad \text{(mean square displacement)}$$

Classical diffusion ($\alpha = 1$): $msd(t) = ct$

Time-fractional diffusion: $msd(t) = ct^{\alpha} \quad (\alpha \in (0,1))$

Ultraslow-diffusion: $msd(t) \sim c \log t \ (t \to \infty)$. (Kochubei 2008)

see also Metzler,Klafter (2000).

In the general case: $msd(t) = 2d(1 * l)(t)$.

• dynamic processes in materials with memory

• bifractional case: CTRW models with long rests and long jumps

Abstract Volterra equations

View $A := -L$ as an operator on a Banach space X, e.g. $X = L_q(\Omega)$. Convolving (1) with l gives

$$u(t) + (l * Au)(t) = u_0 + (l * f)(t) =: h(t), \quad t \geq 0,$$

where $u : \mathbb{R}_+ \to X$.

Known: abstract theory that generalizes semigroups (existence, uniqueness, regularity, subordination, long-time behaviour...), see e.g. the monograph by Prüss (1993) and Clément, Grimmer, Gripenberg, Londen, Nohel, Z.

Example: $X = L_q(\mathbb{R}^d)$, $1 < q < \infty$, $A = (-\Delta)^{\beta/2}$, $\beta \in (0, 2]$ with $D(A) = H_q^\beta(\mathbb{R}^d)$, $l = g_\alpha$ with $\alpha \in (0, 1]$. $\hookrightarrow$ bifractional case

Optimal L_p-L_q-estimates, strong solutions

Theorem: Let $J = [0, T]$, $1 < p, q < \infty$, $\alpha \in (1/p, 1]$, and $\beta \in (0, 2]$. Then the problem

$$\partial_t^\alpha (u - u_0) + (-\Delta)^{\beta/2} u = f, \;\; t \in J, \, x \in \mathbb{R}^d,$$
$$u|_{t=0} = u_0, \;\; x \in \mathbb{R}^d$$

admits a unique solution in the space

$$H_p^\alpha(J; L_q(\mathbb{R}^d)) \cap L_p(J; H_q^\beta(\mathbb{R}^d))$$

if and only if

$$f \in L_p(J; L_q(\mathbb{R}^d)) \quad \text{and} \quad u_0 \in B_{qp}^{\beta(1-\frac{1}{p\alpha})}(\mathbb{R}^d).$$

Proof: Follows from abstract results on maximal L_p-regularity, which can be proved by operator-theoretic and Laplace transform methods, see e.g. Bajlekova (2001) and Z. (PhD thesis 2003 and JEE 2005, more general results including b.c.).

Weak solutions in an abstract Hilbert space setting

Theorem: (Z., FE 2009) Let V, H be separable Hilbert spaces with $V \hookrightarrow H$ densely; identify H with H'. Suppose that $a(t,\cdot,\cdot) : V \times V \to \mathbb{R}$ is bilinear for a.a. $t \in J = [0,T]$, and that $\exists M, c, d > 0$ such that

$$|a(t,u,v)| \leq M|u|_V|v|_V \quad \text{and} \quad a(t,u,v) \geq c|u|_V^2 - d|u|_H^2$$

for $u, v \in V$ and a.a. $t \in (0,T)$. Assume that $x \in H$, $f \in L_2(J;V')$, and that $(k,l) \in \mathcal{PC}$. Then the problem

$$\frac{d}{dt}\Big([k*(u-x)](t), v\Big)_H + a\big(t, u(t), v\big) = \big\langle f(t), v\big\rangle_{V'\times V}, \quad v \in V, \text{ a.a. } t \in J$$

has a unique solution u in the space

$$Z := \{u \in L_2(J;V) : \; k*(u-x) \in {}_0H_2^1(J;V')\}.$$

Remark: Note that $u \in Z$ implies $k * u \in C(J;H)$. Meaning of initial condition is in general not clear. However, if e.g. u and $\frac{d}{dt}(k*[u-x])$ belong to $C([0,T];V')$, then $u(0) = x$.

Example: Take $H = L_2(\mathbb{R}^d)$, $V = H_2^{\beta/2}(\mathbb{R}^d)$, $\beta \in (0, 2)$, and

$$a(t, u, v) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{K(t, x, y)\Big(u(x) - u(y)\Big)\Big(v(x) - v(y)\Big)}{|x - y|^{d+\beta}} \, dx \, dy,$$

where K is merely measurable and $0 < \lambda \leq K \leq \Lambda$. Take further $k = g_{1-\alpha}$, $\alpha \in (0, 1)$. $\hookrightarrow$ generalized bifractional equation in variational form.

Proof: By means of the Galerkin method, the basic energy inequality for the operator $\partial_t(k*)$, and suitable time regularization (Yosida approximation).

Remark: Allen, Caffarelli, Vasseur (ARMA, 2016): existence of weak solutions under stronger assumptions on data (at least L_∞), De Giorgi-Nash-Moser estimates (Hölder regularity)

Decay estimates

For $\mu \geq 0$ define the relaxation function $s_\mu : [0,\infty) \to \mathbb{R}$ via

$$\frac{d}{dt}\Big(k * [s_\mu - 1]\Big) + \mu s_\mu = 0, \;\; t > 0, \quad s_\mu(0) = 1.$$

$(k,l) \in \mathcal{PC} \Rightarrow s_\mu \geq 0, \searrow, \in H^1_{1,\,loc}([0,\infty))$, and $\partial_\mu s_\mu(t) \leq 0$ (see e.g. Prüss 1993).

Let $\Omega \subset \mathbb{R}^d$ be bounded. Consider

$$\partial_t(k * [u - u_0]) - \Delta u = 0, \; t > 0, \; x \in \Omega,$$

with hom. Dirichlet b.c. and $u|_{t=0} = u_0$ in Ω. Let $u_0 \in L_2(\Omega)$ and $\{\phi_n\}_{n=1}^\infty \subset {}_0H^1_2(\Omega)$ be the complete set of eigenfunctions of $(-\Delta)_D$ with eigenvalues λ_n, $|\phi_n|_2 = 1$, $n \in \mathbb{N}$. $\lambda_1 :=$ smallest eigenvalue. Then

$$u(t,x) = \sum_{n=1}^{\infty} s_{\lambda_n}(t)(\phi_n|u_0)\phi_n(x), \quad t > 0, \, x \in \Omega.$$

$$\Rightarrow \quad |u(t,\cdot)|_2 \leq s_{\lambda_1}(t)|u_0|_2, \quad t > 0.$$ (optimal decay estimate)

This extends, e.g., to the case with $(-\Delta_D)^{\beta/2}$, $\beta \in (0,2)$.

Examples: **(i)** $k = g_{1-\alpha}$, $\alpha \in (0,1)$:

$$s_\mu(t) = E_\alpha(-\mu t^\alpha), \quad E_\alpha(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + 1)},\ z \in \mathbb{C}.$$

(Mittag-Leffler function) $\exists c_i(\alpha) > 0$ such that $\frac{c_1}{1+x} \le E_\alpha(-x) \le \frac{c_2}{1+x},\ x \ge 0.$

$$\Rightarrow \quad |u(t,\cdot)|_2 \le \frac{C(\alpha,\lambda_1)}{1+t^\alpha}|u_0|_2, \quad t > 0. \quad \text{(algebraic decay)}$$

(ii) $k(t) = \int_0^1 g_\beta(t)\, d\beta$: $s_\mu(t) \sim C(\log t)^{-1},\ t \to \infty$. (Kochubei 2008)

$$\Rightarrow \quad |u(t,\cdot)|_2 \le \frac{C}{\log t}|u_0|_2, \quad t \ge 2. \quad \text{(logarithmic decay)}$$

(iii) $k(t) = g_{1-\alpha}(t)e^{-\gamma t}$, $l(t) = g_\alpha(t)e^{-\gamma t} + \gamma(1 * [g_\alpha e^{-\gamma \cdot}])(t)$, $\alpha \in (0,1)$, $\gamma > 0$: $s_\mu(t) \le Ce^{-\omega t}$, $t > 0$ for some $\omega(\alpha,\gamma,\mu) \in (0,\gamma)$.

$$\Rightarrow \quad |u(t,\cdot)|_2 \le Ce^{-\omega t}|u_0|_2, \quad t > 0. \quad \text{(exponential decay)}$$

(iv) $k(t) = g_\alpha(t)e^{-\gamma t} + \gamma(1 * [g_\alpha e^{-\gamma \cdot}])(t)$, $l(t) = g_{1-\alpha}(t)e^{-\gamma t}$, $\alpha \in (0,1)$, $\gamma > 0$: $s_\mu(t) \to (1 + \mu |l|_{L_1(\mathbb{R}_+)})^{-1} > 0$, $t \to \infty$. $\Rightarrow$ no decay!

Remark: If $(k,l) \in \mathcal{PC}$ then s_μ cannot decay faster than k.

The case of a bounded domain in the weak setting

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. Consider

$$\begin{aligned} \partial_t\Big(k * [u - u_0]\Big) - \mathrm{div}\,\Big(A(t,x)\nabla u\Big) &= 0, \quad t > 0,\, x \in \Omega, \\ u &= 0, \quad t > 0,\, x \in \partial\Omega, \\ u|_{t=0} &= u_0, \quad x \in \Omega. \end{aligned} \tag{2}$$

Let $u_0 \in L_2(\Omega)$ and assume $A \in L_\infty((0,T) \times \Omega; \mathbb{R}^{d\times d})$ for all $T > 0$ and $\exists \nu > 0$ such that $A(t,x) \geq \nu I$ for a.a. $t \in (0,\infty)$ and $x \in \Omega$.

Theorem: (VZ, SIMA, 2015) Let $(k,l) \in \mathcal{PC}$. Under the above assumptions there holds

$$|u(t,\cdot)|_2 \leq s_{\nu\lambda_1}(t)|u_0|_2, \quad \text{for a.a. } t > 0. \tag{3}$$

(3) is optimal as the special case $A = I$ shows. Generalizations, e.g., to quasilinear problems of the form

$$\partial_t\Big(k * [u - u_0]\Big) - \mathrm{div}\, A(t,x,u,\nabla u) = 0$$

with suitable structure conditions on A.

Bifractional diffusion equations in $\mathbb{R}^d$

Let $\alpha \in (0,1)$ and $\beta \in (0,2]$. Consider

(4) $\quad \partial_t^\alpha (u - u_0) + (-\Delta)^{\beta/2} u = 0$ in $(0,\infty) \times \mathbb{R}^d, \quad u|_{t=0} = u_0$ in $\mathbb{R}^d$.

Let $Z(t,x)$ be the fundamental solution. We have (see e.g. KSZ, 2015):

$$Z(t,x) = \pi^{-d/2} |x|^{-d} H_{32}^{12}\left(2^\beta t^\alpha |x|^{-\beta} \,\middle|\, \begin{matrix} (1-d/2,\beta/2), (0,1), (0,\beta) \\ (0,1), (0,\alpha) \end{matrix} \right).$$

Set $R = t^{-\alpha}|x|^\beta$. Then

$$\begin{aligned} Z(t,x) &\le C t^\alpha |x|^{-d-\beta}, && \text{if } R \ge 1, \\ Z(t,x) &\le C t^{-\alpha} |x|^{-d+\beta}, && \text{if } R \le 1 \text{ and } d > \beta, \\ Z(t,x) &\le C t^{-\alpha} \left(|\log R| + 1\right), && \text{if } R \le 1 \text{ and } d = \beta, \\ Z(t,x) &\le C t^{-\frac{\alpha d}{\beta}}, && \text{if } R \le 1 \text{ and } d < \beta, \end{aligned}$$

$Z(t,\cdot)$ is a pdf for all $t > 0$.

Set $\kappa(\beta, d) := \frac{d}{d-\beta}$, $d > \beta$, and $\kappa(\beta, d) := \infty$, otherwise.

Theorem: (KSVZ, 2014; KSZ, 2015) (i) Let $d \in \mathbb{N}$ and $t > 0$. Then $Z(t, \cdot)$ belongs to $L_p(\mathbb{R}^d)$ if $1 \le p < \kappa(\beta, d)$, and

$$|Z(t, \cdot)|_p \lesssim t^{-\frac{\alpha d}{\beta}\left(1-\frac{1}{p}\right)}, \quad t > 0.$$

(ii) Let $d > \beta$ and $t > 0$. Then $Z(t, \cdot) \in L_{\frac{d}{d-\beta}, \infty}(\mathbb{R}^d)$ and

$$|Z(t, \cdot)|_{\frac{d}{d-\beta}, \infty} \lesssim t^{-\alpha}, \quad t > 0.$$

Cor.: Let $u_0 \in L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$ and $u(t, x) := \int_{\mathbb{R}^d} Z(t, x-y) u_0(y)\, dy$. Then for all $d \in \mathbb{N} \setminus \{2\beta\}$, we have

$$|u(t, \cdot)|_2 \lesssim t^{-\alpha \min\{1, \frac{d}{2\beta}\}}.$$

In case $d = 2\beta$ we have $|u(t, \cdot)|_{2,\infty} \lesssim t^{-\alpha}$.

Remark: In the case $\alpha = 1$: $|u(t, \cdot)|_2 \lesssim t^{-\frac{d}{2\beta}}$ for all $d \in \mathbb{N}$.

Large time behaviour of $Z(t,\cdot) \star u_0$

Assume $\beta \in (1,2]$. Set $\kappa_1(\beta,d) := \frac{d}{d-\beta+1}$, $d > \beta - 1$, and $\kappa_1(2,1) := \infty$.

Theorem: Let $d \in \mathbb{N}$ and $1 \leq p < \kappa_1(\beta,d)$. Let $u_0 \in L_1(\mathbb{R}^d)$ and set $M = \int_{\mathbb{R}^d} u_0(y)\, dy$.

(i) There holds

$$t^{\frac{\alpha d}{\beta}\left(1-\frac{1}{p}\right)} |u(t) - MZ(t)|_p \to 0, \quad \text{as } t \to \infty.$$

(ii) Assume in addition that $||x|u_0|_1 < \infty$. Then

$$t^{\frac{\alpha d}{\beta}\left(1-\frac{1}{p}\right)} |u(t) - MZ(t)|_p \lesssim t^{-\frac{\alpha}{\beta}}, \quad t > 0.$$

Moreover, in the limit case $p = \kappa_1(\beta,d)$ we have

$$t^{\frac{\alpha(\beta-1)}{\beta}} |u(t) - MZ(t)|_{\kappa_1(\beta,d),\infty} \lesssim t^{-\frac{\alpha}{\beta}}, \quad t > 0.$$

Remark: The proof uses bounds for ∇Z, which is only integrable for $\beta \in (1,2]$.

The general case with Laplacian in $\mathbb{R}^d$

Let $(k,l) \in \mathcal{PC}$ and consider

$$\partial_t\Big(k * [u - u_0]\Big) - \Delta u = 0 \text{ in } (0,\infty)\times\Omega, \quad u|_{t=0} = u_0 \text{ in } \mathbb{R}^d.$$

Theorem: (KSVZ, 2014) Let $u(t,\cdot) = Z(t,\cdot) \star u_0$.

(i) Let $d \in \mathbb{N}$, $1 < p < \kappa(d)$, $1 < q, r < \infty$, $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, and $u_0 \in L_q(\mathbb{R}^d)$. Then

$$|u(t)|_r \lesssim \Big[(1 * l)(t)\Big]^{-\frac{d}{2}\left(1-\frac{1}{p}\right)}, \quad t > 0.$$

(ii) Let $d \geq 3$, $1 < q, r < \infty$, $\frac{1}{r} + \frac{2}{d} = \frac{1}{q}$, and $u_0 \in L_q(\mathbb{R}^d)$. Then

$$|u(t)|_r \lesssim \Big[(1 * l)(t)\Big]^{-1}, \quad t > 0.$$

(iii) Let $d \geq 3$ and $u_0 \in L_1(\mathbb{R}^d)$. Then

$$|u(t)|_{\frac{d}{d-2},\infty} \lesssim \Big[(1 * l)(t)\Big]^{-1}, \quad t > 0.$$

Proof: Uses the Fourier transform, appropriate estimates for the relaxation function s_μ and Mihlin's multiplier theorem.

Harnack estimates

Let $\alpha \in (0,1)$, $\Omega_T = (0,T) \times \Omega$, and consider

(5) $$\partial_t^\alpha (u - u_0) - \text{div}\,(A(t,x)\,\nabla u) = 0, \quad (t,x) \in \Omega_T.$$

Assume $A \in L_\infty(\Omega_T; \mathbb{R}^{d\times d})$ and $\exists \nu > 0$ s.t. $A(t,x) \geq \nu I$ a.e.

There holds a weak Harnack inequality for positive weak supersolutions of (5) with critical exponent $p_* = \frac{2+\alpha d}{2+\alpha d - 2\alpha}$ (Z., Ann. Sc. NSP 2013).

$$Q_- := Q_-(t_0, x_0, r) = (t_0, t_0 + r^{2/\alpha}) \times B(x_0, r),$$
$$Q_+ := Q_+(t_0, x_0, r) = (t_0 + 2r^{2/\alpha}, t_0 + 3r^{2/\alpha}) \times B(x_0, r).$$

Suppose $u \geq 0$ in $(0, t_0 + 3r^{2/\alpha}) \times B(x_0, 2r) \subset \Omega_T$ and u is a weak supersolution in $(t_0, t_0 + 3r^{2/\alpha}) \times B(x_0, 2r)$, $u_0 \geq 0$. Let $p \in (0, p_*)$. Then

$$\Big(\frac{1}{|Q_-|}\int_{Q_-} u^p \, d\lambda_{d+1}\Big)^{1/p} \leq C \operatorname*{ess\,inf}_{Q_+} u,$$

where C is independent of u, r, t_0, x_0.

Also known: Hölder continuity of weak solutions (Z., Math. Ann. 2013)

Question: Does the full Harnack inequality (scale invariant) hold for local solutions which are globally positive? Even unknown for $A = I$.

Note that such a full Harnack inequality does hold in the purely time-dependent case! (Z. 2011)

Failure of full Harnack

Theorem: (DKSZ, 2016) Let $\alpha \in (0,1)$ and $\beta \in (0,2]$. Then the full Harnack inequality fails to hold for positive solutions of

(6) $$\partial_t^\alpha (u - u|_{t=0}) + (-\Delta)^{\beta/2} u = 0, \quad t > 0,\ x \in \mathbb{R}^d,$$

in the following cases: (i) $\beta \in (1,2]$ and $d \geq 2$, (ii) $\beta \in (0,1)$ and $d \geq 1$.

There exists a sequence (u_n) of smooth positive solutions of (6) in $(0,\infty)\times\mathbb{R}^d$ s.t. $\forall t_1, t_2 > 0$ and $\forall x_0 \in \mathbb{R}^d \setminus \{0\}$ we have

$$\frac{u_n(t_1,0)}{u_n(t_2,x_0)} \to \infty \quad \text{as } n \to \infty.$$

We can also construct u_n such that

$$\frac{u_n(t_1,0)}{|u_n(t_2,\cdot)|_{L_p(\mathbb{R}^d)}} \to \infty \quad \text{as } n \to \infty,$$

if $1 \leq p < \frac{d}{\beta}$, and if $p = 1$ and $d = \beta$. $\Rightarrow$ Failure of $\sup_{Q_-} u \leq C|u|_{L_p(Q_+)}$!

Remark: The case $\beta \in (1,2]$ and $d = 1$ remains open.

A counterexample

Let $u_0(x) = e^{-|x|^2}$ and $u_0^n(x) := n^{d/p} u_0(nx)$. Let $u_n(t,\cdot) := Z(t,\cdot) \star u_0^n$ ($\star$ convolution in $\mathbb{R}^d$). Then for fixed $t > 0$, and for $p \geq 1$,

$$\begin{aligned} u_n(t,0) &= \mathcal{F}^{-1}\Big(E_\alpha(-|\xi|^\beta t^\alpha)\widehat{u_0^n}(\xi)\Big)(0) \\ &= (2\pi)^{-d} n^{\frac{d}{p}-d} \int_{\mathbb{R}^d} E_\alpha(-|\xi|^\beta t^\alpha)\widehat{u_0}(\xi/n)\, d\xi \\ &= (2\pi)^{-d} n^{\frac{d}{p}} \int_{\mathbb{R}^d} E_\alpha(-|n\xi|^\beta t^\alpha)\widehat{u_0}(\xi)\, d\xi. \end{aligned}$$

Recall $E_\alpha(-\rho) \geq \frac{c_1}{1+\rho}$ for all $\rho \geq 0$. $\widehat{u_0}$ is a Gaussian, thus with $\delta = t^{-\alpha/\beta}$ we have $\widehat{u_0} > 0$ in $B(0,2\delta)$, and so with some $c = c(t,\alpha,d)$,

$$u_n(t,0) \geq c n^{\frac{d}{p}} \int_{B(0,2\delta)} \frac{d\xi}{1+|n\xi|^\beta t^\alpha} \geq \tilde{c} n^{\frac{d}{p}-\beta} \int_{\frac{\delta}{n}}^{2\delta} r^{d-1-\beta} dr.$$

$$\Rightarrow \quad u_n(t,0) \gtrsim n^{\frac{d}{p}-\beta} \text{ if } d > \beta \quad \text{and } u_n(t,0) \gtrsim n^{\frac{\beta}{p}-\beta} \log(n) \text{ if } d = \beta.$$

Hence $u_n(t,0) \to \infty$ if (i) $p < \frac{d}{\beta}$ and if (ii) $p = 1$ and $d = \beta$. But by Young

$$|u_n(t,\cdot)|_{L_p(\mathbb{R}^d)} \leq |Z(t,\cdot)|_{L_1(\mathbb{R}^d)} |u_0^n|_{L_p(\mathbb{R}^d)} = |u_0|_{L_p(\mathbb{R}^d)} < \infty.$$

A positive Harnack result

Theorem: (DKSZ, 2016) Let $\alpha \in (0,1)$, $\beta = 2$, $d \geq 3$, $u(t,\cdot) = Z(t,\cdot) \star u_0$ with $u_0 \geq 0$ and sufficiently regular. Let $r > 0$, $x_1, x_2 \in B(0,r)$,

$$(2r)^{\frac{2}{\alpha}} \leq t_1 < t_2 \leq t_1 + (2r)^{\frac{2}{\alpha}} \quad \text{and } r_i := 2t_i^{\frac{\alpha}{2}},\ i = 1,2.$$

Then there exists a constant $C = C(d,\alpha,t_2/t_1) > 0$ such that

$$u(t_1,x_1) \leq C\Big[1 + \frac{[G \star (u_0 \chi_{B(x_1,r_1)})](x_1)}{[G \star (u_0 \chi_{B(x_2,r_2)})](x_2)}\Big] u(t_2,x_2),$$

where $G(x) = c|x|^{2-d}$ is the Newtonian kernel. The constant blows up as $t_2 \to t_1$.

Remark: (i) Harnack preserving property: If $\sup_{B(x_1,r_1)} u_0 \leq H_0 \inf_{B(x_2,r_2)} u_0$ then

$$u(t_1,x_1) \leq C(d,\alpha,H_0,t_2/t_1) u(t_2,x_2).$$

(ii) Similar result for $d = 2$. If $d = 1$ Harnack holds without additional term.