Multiple solutions for Dirichlet nonlinear BVPs involving fractional Laplacian

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Introduction

We consider a class of nonlocal BVP:

- involving fractional Laplacian,

We investigate the problems of

- existence and multiplicity of solutions

Harnack type inequality on interval

Suppose we have a concave function with zero values at the boundary

$$-u'' \geq 0, u(0) = u(1) = 0$$

then its interior values can be compared with the maximum

$$4 \inf_{[1/4,3/4]} u \geq \sup_{[0,1]} u.$$

We want to generalize this to fractional one-dimensional laplacian.
We hope to apply it to estimate the inverse operator and get expansion.
Motivation: JMAA'13, result: DCDS-B'14, extension: higher dim in annulus?

Nonlocal Problem Involving Fractional Laplacian

The existence of at least two nonnegative solutions u to the nonlocal BVPs involving fractional laplacian with $\alpha \in (1,2]$, $p > 1$, $d = 1$ ($d = 2$) in the ball

$$(-\Delta)_D^{\alpha/2} u = u^p + h$$

under suitable smallness assumption imposed on h, with respect to the norm of

$$\left((-\Delta)_D^{\alpha/2}\right)^{-1}$$

by the reduction to the integral equation

$$u = \left((-\Delta)_D^{\alpha/2}\right)^{-1} (u^p + h) = Bu + u_0$$

in some appropriate cone for fixed $0 < \delta < 1$ and $0 < \gamma < 1$

$$P = \{u \in C(\overline{B(0,1)}) : u \geq 0, \inf_{\overline{B(0,\delta)}} u \geq \gamma \sup_{\overline{B(0,1)}} u,\ u \text{ symmetric, unimodal}\}$$

Superlinear equations on real line

It is well known that the superlinear equation with $p > 1$ on the real line

$$u = bu^p + u_0$$

can have none, one or more solutions depending on the data $b > 0$ and $u_0 \geq 0$. For example, if we additionally assume that

$$bu_0^{p-1} < c_p$$

for some constant

$$c_p = \left((p-1)^{\frac{1-p}{p}} + (p-1)^{\frac{1}{p}}\right)^{-p}$$

then the existence of at least two nonnegative solutions of the equation is guaranteed if we analyse the minimum of the values $bu^{p-1} + u_0 u^{-1}$ compared to the constant 1 and obtain thus two points of the intersection of these graphs.

Superlinear equations on cone in Banach space

The observation from the previous page can be generalised if we replace power term bu^p defined on the real line with a power like nonlinearity in Banach space under some additional, suitable conditions like: coercivity and compactness on some cone in this Banach space. More specifically, we shall consider the equation in the cone P in the Banach space E with the norm $|\cdot|$ in the form

$$u = Bu + u_0$$

for given $u_0 \in P$ and p-power, coercive and compact form B defined on P. The assumption guaranteeing the existence of at least two solutions for the quadratic equation on the real line has to be adequately rephrased as

$$b|u_0|^{p-1} < c_p$$

where $b > 0$ denotes the best estimate b for any $u \in P$

$$|Bu| \leq b|u|^p$$

Krasnosielskii Theorem on Cone Expansion and Compression

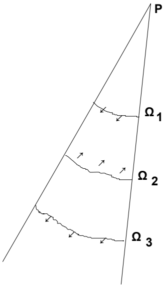
Let E be a Banach space, and let $P \subset E$ be a cone in E. Let Ω_1, Ω_2 be bounded, open sets in E such that $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$. Let completely continuous operator $T : P \to P$ satisfy compression conditions

$$|Tu| \geq |u| \ \text{ for any } \ u \in P \cap \partial\Omega_1 \ \text{ and } \ |Tu| \leq |u| \ \text{ for any } \ u \in P \cap \partial\Omega_2 \,,$$

and the following two expansion conditions for $\overline{\Omega_2} \subset \Omega_2'$ $\overline{\Omega_2'} \subset \Omega_3$

$$|Tu| \leq |u| \ \text{ for any } \ u \in P \cap \partial\Omega_2' \ \text{ and } \ |Tu| \geq |u| \ \text{ for any } \ u \in P \cap \partial\Omega_3$$

are satisfied. Then T has at least two fixed points.



Fractional Laplacian

Fractional laplacian

$$(-\Delta)^{\alpha/2} u(x) = c_{d,-\alpha} \lim_{\varepsilon \to 0} \int_{\{y \in \mathbb{R}^d : |x-y| > \varepsilon\}} \frac{u(x)-u(y)}{|x-y|^{d+\alpha}} dy$$

is well defined for bounded and locally C^2 function u, where

$$c_{d,\gamma} = \Gamma((d-\gamma)/2)/(2^{\gamma} \pi^{d/2} |\Gamma(\gamma/2)|)$$

Alternative definitions:

- weighted second order differential quotient

$$(-\Delta)^{\alpha/2} u(x) = \frac{c_{d,-\alpha}}{2} \int_{R^d} \frac{2u(x)-u(x+y)-u(x-y)}{|y|^{d+\alpha}} dy$$

- via Fourier transform multiplier

$$\widehat{(-\Delta)^{\alpha/2} u} = |\xi|^{\alpha} \hat{u}$$

- limit of the derivative of the solution to some degenerate elliptic equation

Motivation for Fractional Laplacian

- Probability - Math. Finance - infinitesimal generators of stable Lévy processes

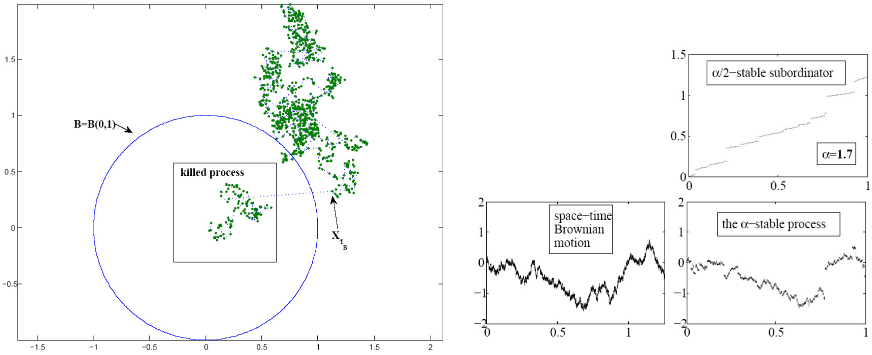


Figure: by K. Bogdan, in: Potential Analysis of Stable Processes and its Extensions

- Mechanics - elastostatics Signorini obstacle problem in linear elasticity
- Fluid Mechanics - quasi-geostrophic fractional Navier-Stokes equation

Instructive linear example

The linear equation

$$(-\Delta)^{\alpha/2} u = 1 \ \text{ in } \ B(0,1)$$

with Dirichlet boundary condition

$$u = 0 \ \text{ in } \ \mathbb{R}^d \setminus B(0,1)$$

posesses solution explicitly defined as

$$u(x) = c^{\alpha,d} \left(1 - |x|^2\right)^{\alpha/2}$$

is of $C^{\alpha/2}$ up to the boundary but not any better, where

$$c^{\alpha,d} = \frac{2^{-\alpha}\Gamma(d/2)}{\Gamma((d+\alpha)/2)\Gamma(1+\alpha/2)}$$

Krylov type estimates on regularity: X. Ros-Oton, J. Serra

For any $g \in L^\infty$ and distance function $\delta(x) = |x - x/|x||$ a solution u of the Dirichlet problem, i.e. it satisfies $u = 0$ in $\mathbb{R}^d \setminus B(0,1)$ together with equation

$$(-\Delta)^{\alpha/2} u = g \ \text{ in } \ B(0,1)$$

can be continuously extended to $u/\delta \in C^{\alpha/2}(\overline{B(0,1)})$ and we control the norm

$$|u/\delta|_\gamma \leq C|g|$$

with $\gamma < \min\{\alpha/2, 1-\alpha/2\}$. Nevertheless, it suffices for compactness of

$$\left((-\Delta)_D^{\alpha/2}\right)^{-1} : C(\overline{B(0,1)}) \to C(\overline{B(0,1)})$$

where this operator is defined by integral operator with Green function kernel as

$$\left((-\Delta)_D^{\alpha/2}\right)^{-1} g(x) = \int_{B(0,1)} G(x,y)g(y)\,dy$$

Green function for the Fractional Laplacian

For any $\alpha \in (0, 2]$ the Green function for the Dirichlet problem in the unit ball

$$G(x,y) = c_\alpha^d |x-y|^{\alpha-d} \int_0^{w(x,y)} r^{\alpha/2-1}(r+1)^{-d/2}\,dr$$

where

$$w(x,y) = (1-|x|^2)(1-|y|^2)|x-y|^{-2}$$

and

$$c_\alpha^d = \Gamma(d/2)/(2^\alpha \pi^{d/2}\Gamma^2(\alpha/2))$$

Green function for $(-1, 1)$ and $\alpha = 0.9$

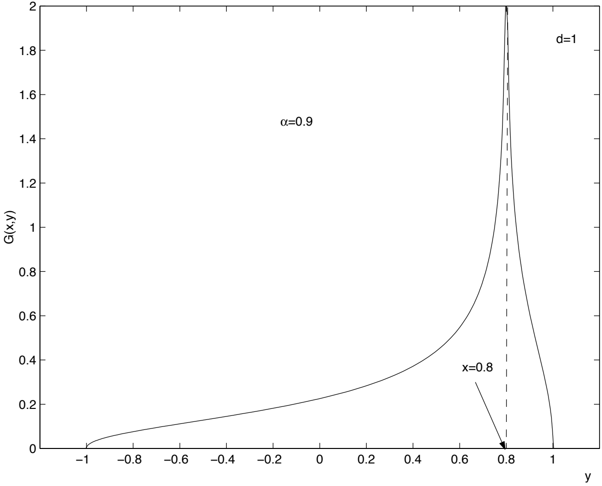


Figure: by K. Bogdan, in: Potential Analysis of Stable Processes and its Extensions

Higher regularity: X. Ros-Oton, J. Serra, X. Cabré, Y. Sire

For any

$$g \in C^{\beta}(B(0,1))$$

the Dirichlet solution to the equation involving fractional laplacian

$$(-\Delta)^{\alpha/2} u = g \text{ in } B(0,1)$$

is of order

$$u \in C^{\beta+\alpha}(B(0,1))$$

Therefore, provided $g \in C^{\gamma}(B(0,1))$ with $\gamma > 2 - \alpha$ our solution is classical $C^2(B(0,1))$ one or even a bit more regular - 2nd derivative Hölder continuous.

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Publication Guide by Mark Kac from:

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FEIGENBAUM: But isn't it nonetheless true that any good mathematician has a very strong conceptual understanding of the things he is working on? He isn't just doing some succession of little proofs.

KAC: Well, the really good ones, yes. But then, you see, there is a gamut, a continuum. In fact, let me put this in because I would like to record it for posterity. I think there are two acts in mathematics. There is the ability to prove and the ability to understand. Now the actions of understanding and of proving are not identical. In fact, it is quite often that you understand something without being able to prove it. Now, of course, the height of happiness is that you understand it and you can prove it. The next stage is that you don't understand it, but you can prove it. That happens over and over again, and mathematics journals are full of such stuff. Then there is the opposite, that is, where you understand it, but you can't prove it. Fortunately, it then may get into a physics journal. Finally comes the ultimate of dismalness, which is in fact the usual situation, when you neither understand it nor can you prove it. The way mathematics is

Thank you for your attention!