Spectral asymptotics

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Theorem (R.B.& T. Carroll-1994)

$D \subset \mathbb{R}^2$ simply connected. λ_1 lowest eigenvalue of Laplacian,
$R_D = \sup_{x \in D} d_D(x)$=inner radius.

$$\frac{0.6197}{R_D^2} < \left(\frac{16c_0^2}{7\zeta(3)}\right)\frac{1}{R_D^2} \leqslant \lambda_1 \leqslant \frac{j_0^2}{R_D^2}$$

- Not true for general D
- Proof related eigenvalue to exit times of BM (well-know) but then related this to hyperbolic geometry, . . .
- (Trivial) Upper bound is sharp
- Lower bound disproved a conjecture of R. Osserman (1977)
- Best lower bound (and extremal domain) unknown.
- 0.6197 remains best known

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Davar Khoshnevisan "Rodrigo: Can you get (such) good (near sharp) estimates for the fractional Laplacian for an interval?" Me: "Are you joking? For an interval? Don't they know that explicitly?"

Could prove many "interesting" (to me) and perhaps "easy" (to others) estimates

Example

- I could get fairly "good" upper and lower estimates on λ_1, Faber-Krahn, isoperimetric inequalities for fractional Laplacian (stable processes) for all $0 < \alpha \leqslant 2$.
- All essentially from multiple integral techniques of Brascamp-Lieb (which I learned from B. Simon in the early 80's) and which I had used many times for problems (spectral gaps, hot-spots, etc) for $\alpha = 2$.

While at it: raised several naive and quite non-imaginative questions/conjectures

- Geometric shape of ground state eigenfunction and expected exit times (Green potentials) for convex domains.
- Spectral gaps estimates . . . .
- Spectral asymptotics
- . . .

H. Weyl's Law (1912), Dirichlet eigenvalues: $N_D(\lambda) = \#\{\lambda_k | \lambda_k < \lambda\}$

$$\lim_{\lambda\to\infty} \lambda^{-d/2} N_D(\lambda) = \frac{|D|}{(4\pi)^{d/2}\Gamma(d/2+1)} = C_d|D|, \quad D \subset \mathbb{R}^d, \quad d \geqslant 2$$

Conjecture (Weyl 1913)

$$N_D(\lambda) = C_d|D|\lambda^{d/2} - C_d'|\partial D|\lambda^{(d-1)/2} + o(\lambda^{(d-1)/2}), \quad \textit{as} \quad \lambda \to \infty,$$

Theorem (Seeley 1978)

$$N_D(\lambda) = C_d|D|\lambda^{d/2} + O(\lambda^{(d-1)/2}), \quad \textit{as} \quad \lambda \to \infty.$$

(Many others: Courant (1922), Levitan (1952), Hörmander (1968) . . .)

Theorem (V. Ivrii 1980)

- *Conjecture is true for "nice" domains, for example, nice smooth convex domains, $\cdots$*

Second proof by Melrose, a little later in 1980.

Weyl (1950) "Ramifications, old and new, of the eigenvalue problem" Bull. AMS

"I feel that these informations about the proper oscillations of a membrane, valuable as they are, are still incomplete. I have certain conjectures on what a complete analysis of their asymptotic behavior should aim at; but since for more than 35 years I have made no serious attempt to prove them, I think I had better keep them to myself."

Weyl passed away Dec. 1955.

Theorem (Minakshiusundaram 1953)

Laplace trasform of Spectral Counting Function.

$$Z_t(D) = \int_D p_t^D(x,x)\,dx = \sum_{k=1}^{\infty} e^{-\lambda_k t} = \int_0^{\infty} e^{-\lambda t} N_D(d\lambda)$$

$$Z_D(t) = \frac{1}{(4\pi t)^{d/2}} \left\{ |D| - \frac{\sqrt{\pi}}{2} |\partial D|\, t^{1/2} + \sum_{j=2}^{N} c_j(D) t^{j/2} + \mathcal{O}(t^{(N+1)/2}) \right\},$$

for domains with smooth boundaries.

Theorem (McKean-Singer 1967)

(After the 1966 celebrated paper "Can one hear the shape of a drum?" of Mark (Marek) Kac)

$D \subset \mathbb{R}^2$ *with* r *holes,*

$$c_2 = \frac{(1-r)}{6}$$

Kac (1951) New proof of Weyl's First Law: "For small time the Dirichlet heat kernel $p_t^D(x, y$ (transitions of killed BM) is the same as the free kernel $p_t(x, y)$"

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The Principle of not feeling the boundary: (Poetically stated in his 1966 paper)

"As the Brownian particles begin to diffuse they are not aware, so to speak, of the disaster that awaits them when they reach the boundary."

The rotationally symmetric α-process, $0 < \alpha < 2$, $X = \left\{X_t, t \geqslant 0, P_x, x \in \mathbb{R}^d\right\}$ is a Lévy process with $E_x(e^{i\xi\cdot(X_t - X_0)}) = e^{-t|\xi|^\alpha}, \quad x \in \mathbb{R}^d, \ \xi \in \mathbb{R}^d$.

Transition probabilities

$$p_t^\alpha(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{-t|\xi|^\alpha}\, d\xi = \int_0^\infty \frac{1}{(4\pi s)^{d/2}} e^{\frac{-|x|^2}{4s}} \eta_t^{\alpha/2}(s)\, ds,$$

$\eta_t^{\alpha/2}(s)$ density for $\alpha/2$-stable subordinator.

$$p_t^\alpha(x) = t^{-d/\alpha} p_1^\alpha(t^{-1/\alpha}x)$$

$$p_t^\alpha(0) = t^{-d/\alpha} p_1^\alpha(0)$$

Dirichlet α-heat kernel: τ_D exit time of X_t from D.

$$\begin{aligned} p_t^{D,\alpha}(x,y) &= p_t^\alpha(x-y) - \mathbb{E}_x\left(\tau_D < t, p_{t-\tau_D}^\alpha(X(\tau_D), y)\right) \\ &= p_t^\alpha(x-y) - r_t(x,y). \end{aligned}$$

$$P_t^{D,\alpha}(x,x) = \sum_{j=1}^{\infty} e^{-t\lambda_j}\varphi_j^2(x)$$

$$\begin{aligned}\sum_{j=1}^{\infty} e^{-t\lambda_j} &= \int_D p_t^{D,\alpha}(x,x)dx = \int_D p_t^{\alpha}(x-x)dx - \int_D r_t(x,x)dx \\ &= p_t^{\alpha}(0)\,|D| - \int_D r_t(x,x)dx = t^{-d/\alpha} p_1^{(\alpha)}(0)|D| - \int_D r_t(x,x)dx\end{aligned}$$

Need a "little" Lemma

$$\lim_{t\to 0} t^{d/\alpha} \int_D r_t(x,x)dx = 0$$

Karamata–tauberian: μ a meaure on $[0,\infty)$
$\lim_{t\to 0} t^{\gamma} \int_0^{\infty} e^{-t\lambda} d\mu(\lambda) = A, \ \Rightarrow \lim_{a\to\infty} a^{-\gamma}\mu[0,a) = \frac{A}{\Gamma(\gamma+1)}, \ \gamma > 0,$

$$\Rightarrow \quad \lim_{\lambda\to\infty} \lambda^{-d/\alpha} N_D(\lambda) = \frac{p_1^{\alpha}(0)|D|}{\Gamma(d/\alpha+1)}$$

[Kac 1951 ($\alpha = 2$), Blumenthal and Getoor 1959 ($0 < \alpha < 2$).]

Second order trace asymtotics

1. R.B., T. Kulczycki, B. Siudeja (2010) (Proved for $\alpha = 2$ by R. Brown in 1993.) D Lipschitz.

$$t^{d/\alpha} Z_D(t) = p_1^{\alpha}(0)\,|D| - p_1^{(\alpha)}(0)\mathcal{L}_{\alpha,d}\,|\partial D| t^{1/\alpha} + o\left(t^{1/\alpha}\right), \;\; t \downarrow 0,$$

2. R.B, J. Mijena, E. Nane (2014): Holds for the **relativistic fractional Laplacian** for smooth domains
3. H. Park, R. Song (2014): Lipschitz domains **relativistic fractional Laplacian**
4. K. Bogdan, B. A. Siudeja (2015) Smooth domains & unimodal Lévy processes with weak scaling property

Question

Minakshiusundaram full expansion? McKean-Singer type third term? **Unknown**

Most interesting (to me) Question:

Does Ivrii hold for fractional Laplacian/rotationally symmetric stable processes?

Conjecture ($D \subset \mathbb{R}^d$, $d \geqslant 2$, smooth)

$$N_D^{(\alpha)}(\lambda) = \frac{p_1^{(\alpha)}(0)}{\Gamma(d/\alpha + 1)} |D| \lambda^{d/\alpha} - \frac{p_1^{(\alpha)}(0)\mathcal{L}_{\alpha,d}}{\Gamma\left(\frac{d-1}{\alpha} + 1\right)} |\partial D| \lambda^{(d-1)/\alpha} + o(\lambda^{(d-1)/\alpha}), \quad \lambda \to \infty$$

$\mathcal{L}_{\alpha,d}$ *is the constant in second order asymptotic for trace.*

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Theorem (Ivrii, 2016)

True for nice smooth domains: arXiv:1603.06364v2 [math.SP] (21 March) 5 April, 2016

Proof based on "extensions of Bouted-de-Mionvel algebra to operators that do not have the transmission property." (**He told me, "full details will be in Section 8.5 of my Monster Book"**)

TO THE YOUNG AND FEARLESS: Read the proof, understand it and explain it to me (rest of us) in a way that I (we) can understand it.

Another "spectral function" whose asymptotic expansion encodes important geometric information for the domain D is the **"Spectral Heat Content"**

$$\begin{aligned} Q_D(t) &= \int_D \int_D p_t^{D,\alpha}(x,y)dydx = \int_D P_x\{\tau_D > t\}dx = \int_D (1 - P_x\{\tau_D \leqslant t\})dx \\ &= |D| - \int_D P_x\{\tau_D \leqslant t\}dx \ \left(= \sum_{k=1}^{\infty} e^{-\lambda_k t} \left(\int_D \varphi_k(x)dx \right)^2 \right) \end{aligned}$$

M. van den Berg& P. Gilkey 1994

$$Q_D(t) = \sum_{j=0}^{N} a_j(D) t^{j/2} + \mathcal{O}(t^{N+1)/2})$$

van den Berg le Gall 1994

$$Q_D(t) = |D| - \frac{2}{\sqrt{\pi}}|\partial D| t^{1/2} + \left(2^{-1}(d-1) \int_{\partial D} H(s)ds \right) t + \mathcal{O}(t^{3/2}),$$

as $t \downarrow 0$. Here, $H(s)$ denotes the mean curvature at the point $s \in \partial D$.

The obvious, naive, questions/conjectures: Replace "2" by "α" and ...

Less obvious and less naive

Conjecture

(i) For $1 < \alpha < 2$, there exists $C_{d,\alpha} > 0$

$$Q_D^{(\alpha)}(t) = |D| - C_{d,\alpha}|\partial D| t^{\frac{1}{\alpha}} + \mathcal{O}(t), \quad t \downarrow 0.$$

(ii) For $\alpha = 1$, there exists $C_d > 0$

$$Q_D^{(\alpha)}(t) = |D| - C_d|\partial D| \, t \ln\left(\frac{1}{t}\right) + \mathcal{O}(t), \quad t \downarrow 0.$$

(iii) For $0 < \alpha < 1$, there exists $C_{d,\alpha} > 0$ such that

$$Q_D^{(\alpha)}(t) = |D| - C_{d,\alpha}\mathcal{P}_\alpha(D) \, t + o(t), \quad t \downarrow 0.$$

Theorem (L. Acuña-Valverde, Purdue 2015 Ph.D. Thesis.) True for $D = (a, b)$.

Theorem (L. Acuña-Valverde, Purdue 2015 Ph.D. Thesis. (J. Geom. Anal. 2016)
$D \subset \mathbb{R}^d$, $d \geqslant 2$, $C^{1,1}$

1. $1 < \alpha < 2$: $\gamma(\alpha) = \Gamma\left(1 - \frac{1}{\alpha}\right)$

$$\frac{1}{\pi}\gamma(\alpha)|\partial D| \leqslant \underline{\lim}_{t\downarrow 0} \frac{|D| - Q_D^\alpha(t)}{t^{\frac{1}{\alpha}}} \leqslant \overline{\lim}_{t\downarrow 0} \frac{|D| - Q_D^\alpha(t)}{t^{\frac{1}{\alpha}}} \leqslant 2^{\frac{3d+1}{2}}\gamma(\alpha)|\partial D|.$$

2. $\alpha = 1$.

$$\frac{1}{\pi}|\partial D| \quad \leqslant \underline{\lim}_{t\downarrow 0} \frac{|D| - Q_D^1(t)}{t\ \ln\left(\frac{1}{t}\right)} \leqslant \overline{\lim}_{t\downarrow 0} \frac{|D| - Q_D^1(t)}{t\ \ln\left(\frac{1}{t}\right)} \leqslant 2^{\frac{3d+1}{2}}|\partial D|.$$

3. $0 < \alpha < 1$: $\mathcal{A}_{\alpha,d} = \alpha\, 2^{\alpha-1}\, \pi^{-1-\frac{d}{2}}\ \sin\left(\frac{\pi\alpha}{2}\right)\ \Gamma\left(\frac{d+\alpha}{2}\right)\ \Gamma\left(\frac{\alpha}{2}\right).$

$$\mathcal{A}_{d,\alpha}\mathcal{P}_\alpha(D) \leqslant \underline{\lim}_{t\downarrow 0} \frac{|D| - Q_D^\alpha(t)}{t} \leqslant \overline{\lim}_{t\downarrow 0} \frac{|D| - Q_D^\alpha(t)}{t} \leqslant C_{d,\alpha}\mathcal{P}_\alpha(D).$$

$\mathcal{P}_\alpha(D) = \int_D \int_{D^c} \frac{dx\ dy}{|x-y|^{d+\alpha}}$ (α-perimeter)

Theorem (Frank-Seiringer 2011 "Non-linear ground state representations and sharp Hardy–inequalities.") There is a constant $C_{d,\alpha} > 0$ s.t.

$$|D|^{(d-\alpha)/d} \leqslant C_{d,\alpha}\mathcal{P}_\alpha(D)$$

with equality if and only if D is a ball.

Theorem (Fusco–Millot–Morini 2012 "A quantitative isoperimetric inequality for fractional perimeters.")

$$\lim_{\alpha\downarrow 0} \alpha\mathcal{P}_\alpha(D) = d\ |B_1(0)|\ |D|$$

$$\lim_{\alpha\uparrow 1} (1-\alpha)\mathcal{P}_\alpha(D) = K_d|\partial D|,$$

for some $K_d > 0$.

Heat content in $\mathbb{R}^d$

$$\mathbb{H}_D^{(\alpha)}(t) = \int_D \int_D p_t^\alpha(x,y)dx\,dy = |D| - \int_D \int_{D^c} p_t^\alpha(x,y)dx\,dy$$
$$= |D| - \mathbb{H}_{D,D^c}^{(\alpha)}(t)$$

Theorem (Preunkert 2004, Miranda-Pallara-Paronetto-Preunker 2007)

For $\alpha = 2$.

1. $$\frac{\mathbb{H}_{D,D^c}^{(2)}(t)}{t^{\frac{1}{2}}} \leqslant \frac{1}{\sqrt{\pi}}|\partial D|, \quad \textit{for all } t > 0.$$

2. $$\lim_{t\downarrow 0} \frac{\mathbb{H}_{D,D^c}^{(2)}(t)}{t^{\frac{1}{2}}} = \frac{1}{\sqrt{\pi}}|\partial D|$$

Theorem (L. Acuña-Valverde (2015))

1. $1 < \alpha < 2$: $\frac{\mathbb{H}^{(\alpha)}_{D,D^c}(t)}{t^{\frac{1}{\alpha}}} \leqslant \frac{1}{\pi}\Gamma\left(1-\frac{1}{\alpha}\right)|\partial D|$, *for all* $t > 0$ *and*

$$\lim_{t\downarrow 0} \frac{\mathbb{H}^{(\alpha)}_{D,D^c}(t)}{t^{\frac{1}{\alpha}}} = \frac{1}{\pi}\Gamma\left(1-\frac{1}{\alpha}\right)|\partial D|.$$

2. *For* $\alpha = 1$:

$$\lim_{t\downarrow 0} \frac{\mathbb{H}^{(1)}_{D,D^c}(t)}{t \ln\left(\frac{1}{t}\right)} = \frac{1}{\pi}|\partial D|,$$

3. *For* $0 < \alpha < 1$:

$$\lim_{t\downarrow 0} \frac{\mathbb{H}^{(\alpha)}_{D,D^c}(t)}{t} = \mathcal{A}_{\alpha,d}\mathcal{P}_{\alpha}(D).$$

Trace and heat content for potentials in $\mathbb{R}^d$

$V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, $H_V = (-\Delta)^{\frac{\alpha}{2}} + V$

$$e^{-tH_V} f(x) = \mathbb{E}^x[e^{-\int_0^t V(X_s)ds} f(X_t)]$$

Feynman-Kac formula

$$p_t^{V,\alpha}(x,y) = p_t^{(\alpha)}(x,y) E_{x,y}^t \left[e^{-\int_0^t V(X_s)ds} \right],$$

(Trace) $$\mathcal{T}_V^{(\alpha)}(t) = \int_{\mathbb{R}^d} \left(p_t^{V,\alpha}(x,x) - p_t^{\alpha}(x,x) \right) dx = p_t^{(\alpha)}(0) \int_{\mathbb{R}^d} E_{x,x}^t \left(e^{-\int_0^t V(X_s)ds} - 1 \right) dx,$$

(Heat Content) $$\mathcal{Q}_{\mathcal{V}}^{(\alpha)}(t) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(p_t^{V,\alpha}(x,y) - p_t^{\alpha}(x,y) \right) dxdy = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_t^{\alpha}(x,y) E_{x,y}^t \left[e^{-\int_0^t V(X_s)ds} - 1 \right] dxdy.$$

$\alpha = 2$: Trace asymptotics well known. Suppose $V \in \mathcal{S}(\mathbb{R}^d)$.

$$\mathcal{T}_V^{(2)}(t) = \frac{1}{(4\pi t)^{d/2}} \left[\sum_{m=1}^{N} C_m(V) t^m + O(t^{N+1}) \right], \quad \text{as} \quad t \downarrow 0.$$

1. $d = 1$, McKean-Moerbeke (1975) give (via KdV) a formula for C_k.
2. $d = 3$, Colin de Verdière (1981) used McKean-Moerbeke techniques and symmetry of some integrals to compute C_1, C_2, C_3, C_4.
3. For all $d \geqslant 1$, R.B.-A. Sá Barreto (1995) a formula is given for C_k in terms of Fourier transforms and C_1, C_2, C_3, C_4, C_5 are computed.

$$(-1)^1 C_1(V) = \int_{\mathbb{R}^d} V(x)dx, \quad (-1)^2 C_2(V) = \frac{1}{2!} \int_{\mathbb{R}^d} V^2(x)dx$$

$$(-1)^3 C_3(V) = \frac{1}{3!} \int_{\mathbb{R}^d} \left(V^3(x)dx + \frac{1}{2} |\nabla V(x)|^2 \right) dx$$

$$(-1)^4 C_4(V) = \frac{1}{4!} \int_{\mathbb{R}^d} \left(V^4(x)dx + 2V(x)|\nabla V(x)|^2 + \frac{1}{5} (\Delta V(x))^2 \right) dx$$

Theorem (R.B. S. Yolcu (2011))

$$\frac{\mathcal{T}_V^{(\alpha)}(t)}{p_t^{(\alpha)}(0)} = -t\int_{\mathbb{R}^d} V(x)dx + \frac{1}{2}t^2\int_{\mathbb{R}^d}|V(x)|^2dx + \mathcal{O}(t^3), \; t \downarrow 0$$

Proof via probabilistic arguments based on Feynman-Kac and estimates on stable bridge

L. Acuña-Valverde (2014, JFA) Found the "general" expansion for $0 < \alpha < 2$–not as clean. From it, one can compute "coefficients."

Example (For $1 < \alpha < 2$, $d \geqslant 3$)

$$\frac{\mathcal{T}_V^{(\alpha)}(t)}{p_t^{(\alpha)}(0)} = -t \int_{\mathbb{R}^d} V(\theta) d\theta + \frac{t^2}{2!} \int_{\mathbb{R}^d} V^2(\theta) d\theta - \left\{ \frac{t^3}{3!} \int_{\mathbb{R}^d} V^3(\theta) d\theta + \mathcal{L}_{d,\alpha}\, t^{2+\frac{2}{\alpha}} \int_{\mathbb{R}^d} |\nabla V(\theta)|^2 \, d\theta \right\} + \mathcal{O}(t^4), \quad t \downarrow 0.$$

$$\mathcal{L}_{d,\alpha} = \frac{C_{d,\alpha} K_1(d,\alpha)}{(2\pi)^d}, \quad C_{d,\alpha} = \frac{\pi^{d/2}}{p_1^{(\alpha)}(0)},$$

$$K_1(d,\alpha) = \int_0^1 \int_0^{\lambda_1} E\left[\frac{S^*_{1-w} S^*_w}{(S^*_{1-w} + S^*_w)^{1+\frac{d}{2}}} \right] dw d\lambda_1.$$

Note: As $\alpha \uparrow 2$, get the 3rd term expansion for the Laplacian.

Example (For $\frac{4}{3} < \alpha < 2$, $d \geqslant 5$)

$$
\begin{aligned}
\frac{\mathcal{T}_V^{(\alpha)}(t)}{p_t^{(\alpha)}(0)} &= -t \int_{\mathbb{R}^d} V(\theta)d\theta + \frac{t^2}{2!} \int_{\mathbb{R}^d} V^2(\theta)d\theta \\
- & \left\{ \frac{t^3}{3!} \int_{\mathbb{R}^d} V^3(\theta)d\theta + \mathcal{L}_{d,\alpha} t^{2+\frac{2}{\alpha}} \int_{\mathbb{R}^d} |\nabla V(\theta)|^2 \right\} \\
+ & \left\{ \frac{t^4}{4!} \int_{\mathbb{R}^d} V^4(\theta)d\theta + \mathcal{M}_{d,\alpha} t^{3+\frac{2}{\alpha}} \int_{\mathbb{R}^d} V(\theta)\, |\nabla V(\theta)|^2 \, d\theta \right. \\
+ & \left. \mathcal{N}_{d,\alpha} t^{2+\frac{2\cdot 2}{\alpha}} \int_{\mathbb{R}^d} |\Delta V(\theta)|^2 \, d\theta \right\} + \mathcal{O}(t^5), \quad t \downarrow 0.
\end{aligned}
$$

NOTE 1: $1 < 2 < 3 < 2 + \frac{2}{\alpha} < 4 < 3 + \frac{2}{\alpha} < 2 + \frac{2\cdot 2}{\alpha} < 5$

NOTE 2: As $\alpha \uparrow 2$, get the 4th term expansion for the Laplacian.

Theorem (L. Acuña-Velverde & R.B. 2015)

For any integer $N \geqslant 2$,

$$Q_V^{(\alpha)}(t) = -t \int_{\mathbb{R}^d} V(\theta) d\theta + \sum_{\ell=2}^{N} (-t)^\ell C_\ell(V) + \mathcal{O}(t^{N+1}), \quad (1)$$

as $t \downarrow 0$. Here,

$$C_\ell(V) = \sum_{\substack{n+k=\ell \\ 0\leqslant n,\, 2\leqslant k}} \frac{1}{n!} C_{n,k}(V),$$

Example

$$Q_V^{(\alpha)}(t) = -t \int_{\mathbb{R}^d} V(\theta) d\theta + \frac{t^2}{2!} \int_{\mathbb{R}^d} V^2(\theta) d\theta - \frac{t^3}{3!} \left(\int_{\mathbb{R}^d} V^3(\theta) d\theta + \mathcal{E}_\alpha(V) \right)$$
$$+ \frac{t^4}{4!} \int_{\mathbb{R}^d} \left(V^4(\theta) d\theta + 2V^2(\theta)(-\Delta)^{\frac{\alpha}{2}} V(\theta) d\theta + \left|(-\Delta)^{\frac{\alpha}{2}} V(\theta)\right|^2 \right) d\theta + \mathcal{O}(t^5),$$

as $t \downarrow 0$. Here, $\mathcal{E}_\alpha$ is the Dirichlet form of $(-\Delta)^{\frac{\alpha}{2}}$

Last (for now) question:

What am I trying to do and why?

Thank you!