A min-max formula for operators that have the global comparison principle (i.e. elliptic operators)

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Collaboration

This is joint work with Nestor Guillen

Warm-up Discussion

What are your top 5 favorite ways to define the $(1/2)$-Laplacian (or s-Laplacian)?

Warm-up Discussion

Assume ϕ is $C^{1,\gamma}(\mathbb{R}^d)$ and U_ϕ is the unique bounded solution of

$$\begin{cases} \Delta U_\phi = 0 & \text{in } \mathbb{R}^{d+1}_+ \\ U_\phi = \phi & \text{on } \mathbb{R}^d. \end{cases}$$

It is well known that

$$\phi \mapsto \partial_n U_\phi = -(-\Delta)^{1/2}\phi \text{ on } \mathbb{R}^d$$

$$-(-\Delta)^{1/2}\phi(x) = p.v. \int_{\mathbb{R}^d} (u(x+h) - u(x)) \, |h|^{-d-1} \, dh$$

Warm-up Discussion

Let $\Omega \subset \mathbb{R}^d$. Given $\phi \in C^{1,\alpha}(\partial\Omega)$ $(\alpha > 0)$, let U_ϕ solve

$$\begin{cases} F(D^2 U_\phi, x) = 0 & \text{in } \Omega \\ U_\phi = \phi & \text{on } \partial\Omega \end{cases}$$

where F is a **uniformly elliptic operator** and U_ϕ is the unique viscosity solution. Set

$$I(\phi, x) := \partial_n U_\phi(x)$$

$$(\ n = \text{inner normal to } \partial\Omega)$$

$$I(\phi, x) \text{``} = \text{''} \text{nonlinear version } \int_{\partial\Omega} (u(y) - u(x)) \, |y - x|^{-d-1} \, d\sigma(h)??????$$

Warm-up Discussion

How many papers can you list that start with a fractional problem e.g. involving $-(-\Delta)^s$ and resolve it by extending to an extra dimension?

How many papers can you list that start with a second order problem and resolve it by fractional (or integro-differential) techniques in one LESS dimension?

Linear Lévy Operator

$L : C^2 \to L^\infty$, with $u \in C^2$

$$L(u,x) = \overbrace{\mathrm{Tr}(A(x)D^2u) + B(x)\cdot\nabla u + C(x)u}^{L_{loc}(u,x)} + \underbrace{\int_{\mathbb{R}^d} (u(x+y) - u(x) - y\cdot\nabla u(x)\mathbb{1}_{B_1}(y))d\mu_x(y)}_{L_{ID}(u,x)}$$

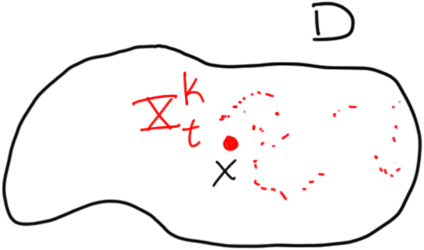
where $A(x) \geq 0$, $C(x) \leq 0$, all of A, B, C are bounded, and μ_x satisfies

$$\int_{\mathbb{R}^d} \min(|y|^2, 1)d\mu_x(y) < +\infty.$$

Linear Lévy Operator

Generators of Markov processes on $\mathbb{R}^d$

$$L(u,x) := \lim_{t\to 0} \frac{\mathbb{E}_x(u(X_t^x(\cdot))) - u(x)}{t}$$



The Global Comparison Principle

Definition

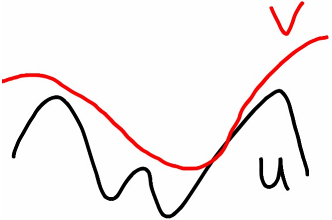
$$I : D \subset \mathbb{R}^X \to \mathbb{R}^X$$

is said to have the **global comparison property** *(GCP) if*

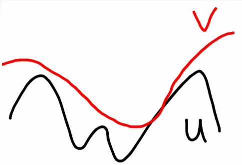
$$u, v \in D \text{ and } v \text{ touches } u \text{ from above at } x_0 \Rightarrow I(u, x_0) \leq I(v, x_0)$$

$$u(x) \leq v(x) \ \forall x \in X$$

$$u(x_0) = v(x_0)$$



Examples of the GCP



Consider $X = \mathbb{R}^d$, $D = C^2(\mathbb{R}^d)$

- $I(f,x) = \Delta f(x)$
- $I(f,x) = \mathrm{Tr}(A(x)D^2 f(x))) + B(x)\cdot\nabla f,\ \ A(x) \geq 0$
- $I(f,x) = F(D^2 f(x))$, $F : \mathrm{Sym}(\mathbb{R}^d) \to \mathbb{R}$ monotone nondec.
- $I(f,x) = H(\nabla f(x)),\ \ H : \mathbb{R}^d \to \mathbb{R}^d$, $D = C^1(\mathbb{R}^d))$
- $I(f,x) = f(x+y) - f(x)$, $y \in \mathbb{R}^d$ fixed, $(D = C^0(\mathbb{R}^d))$
- $I(f,x) = \int f(x+y) - f(x) d\mu(y)$, μ Borel measure $(D = C^0(\mathbb{R}^d))$
- $I(f,x) = -(-\Delta)^s f(x)$, $s \in [0,1]$, $(D = \mathcal{S}(\mathbb{R}^d))$, $d\mu(y) = |y|^{-d-2s}$

Examples of the GCP

Dirichlet to Neumann map (D-to-N) for nonlinear elliptic equations

Let $X = \partial\Omega$, $\Omega \subset \mathbb{R}^d$. Given $\phi \in C^{1,\alpha}(\partial\Omega)$ ($\alpha > 0$), let U_ϕ solve

$$\begin{cases} F(D^2 U_\phi, x) = 0 & \text{in } \Omega \\ U_\phi = \phi & \text{on } \partial\Omega \end{cases}$$

where F is a **uniformly elliptic operator** and U_ϕ is the unique viscosity solution. Set

$$I(\phi, x) := \partial_n U_\phi(x)$$

$$(\ n = \text{inner normal to } \partial\Omega)$$

The map $I : C^{1,\alpha}(\partial\Omega) \to C(\partial\Omega)$ is Lipschitz and has the GCP.

Linear GCP Characterization

Theorem (Philippe Courrège 1965)

Suppose L is an operator for which

1. $L : C_0^2(\mathbb{R}^d) \to C(\mathbb{R}^d)$
2. *L is linear*
3. *L satisfies the GCP.*

Then L is necessarily a linear Lévy operator,

$$L(u,x) = \mathrm{Tr}(A(x)D^2u) + B(x)\cdot\nabla u + C(x)u + \int_{\mathbb{R}^d}(u(x+y) - u(x) - y\cdot\nabla u(x)\mathbb{1}_{B_1}(y))d\mu_x(y)$$

where $A(x) \geq 0$, $C(x) \leq 0$, all of A, B, C are bounded, and μ_x satisfies

$$\int_{\mathbb{R}^d}\min(y^2,1)d\mu_x(y) < +\infty.$$

Lipschitz maps with the GCP

Many of the interesting examples, even in the preceding discussion were not linear– especially the D-to-N. Can you still prove a similar characterization as what Courrège proved for linear operators?

Yes, if you assume I is Lipschitz

GCP is a lot of structure!

For example...

$$|I(u,x) - I(v,x)| \leq$$
$$C(R)\|I\|_{Lip(C_b^\beta, C_b)}(\|u - v\|_{C^\beta(B_R(x))} + \|u - v\|_{L^\infty(M)}).$$

A new result

Theorem (Guillen-Schwab, arXiv two days ago!)

Let M be a complete, d-dimensional manifold, and that $I : C^2(M) \to C^0(M)$ is Lipschitz, with the GCP. Then

$$I(u,x) = \min_a \max_b \{f_{ab}(x) + L_{ab}(u,x)\} \;\; \forall \; u, x.$$

where, for each pair of indices ab, we have

- $f_{ab}(x) \in C^0(\mathbb{R}^d)$ *(uniformly)*
- $L_{ab} : C^2(\mathbb{R}^d) \to C^0(\mathbb{R})$ *is a (uniformly) bounded linear Lévy operator*

A new result

Theorem (Guillen-Schwab, arXiv two days ago!)

Furthermore if $I : C^{1,\gamma}(\mathbb{R}^d) \to C(\mathbb{R}^d)$ is Lipschitz and satisfies the GCP, then

$$\begin{aligned} L(u,x) &= C(x)u(x) + B(x)\cdot\nabla u \\ &\quad + \int_{\mathbb{R}^d} u(x+y) - u(x) - \nabla u(x)\cdot y\chi_{B_1(0)}\, d\mu_x(y) \end{aligned}$$

and

$$\int \min\{|y|^{1+\gamma}, 1\}\, d\mu_x(y) < \infty.$$

A new result

Theorem (Guillen-Schwab, arXiv two days ago!)

Furthermore if $I : C^{0,\gamma}(\mathbb{R}^d) \to C(\mathbb{R}^d)$ is Lipschitz and satisfies the GCP, then

$$L(u,x) = \int_{\mathbb{R}^d} u(x+y) - u(x) - \nabla u(x) \cdot y \chi_{B_1(0)} \, d\mu_x(y)$$

and

$$\int \min\{|y|^\gamma, 1\} \, d\mu_x(y) < \infty.$$

The importance of min-max formulas

For local elliptic equations

$$F(D^2u, \nabla u, u, x) = 0$$

we have that the right hand side can be represented as

$$\min_i \max_j \{f_{ij}(x) + c_{ij}(x)u(x) + \nabla u \cdot b_{ij}(x) + \operatorname{tr}(A_{ij}(x)D^2u(x))\}$$

In this setting, min-max formulas have been of great use, as they allow us to represent solutions to a PDE as the value functions of zero-sum differential games, i.e.

Fully nonlinear elliptic equation $\Leftrightarrow$ Isaacs equation for some game

The importance of min-max formulas

Tools + Problems in PDEs	$\Leftrightarrow$	Tools + Problems in Differential Games

Evans (1984), Souganidis (1985), and Katsoulakis (1995) have used such representions to derive regularity estimates crucial to existence and uniqueness for viscosity solutions of first and second order Hamilton-Jacobi-Bellman equations.

The importance of min-max formulas

Min-max formulas are of great use, as they allow us to represent solutions to a PDE to the value function of a zero-sum differential game.

Tools + Problems in PDEs	$\Leftrightarrow$	Tools + Problems in Differential Games

They have also played a crucial role in work of Kuo-Trudinger (1992) on numerical schemes for uniformly elliptic (fully nonlinear) second order equations.

The importance of min-max formulas

The original motivation for us was twofold:

1) Resolving a standing issue in the integro-differential literature: what is a reasonably wide common framework for nonlocal equations? Many existing results assume the operator is given as a min-max– can we justify it?
Recent years have seen a great deal of progress in the existence/uniqueness theory: Barles-Imbert, Jakobsen-Karlsen, Barles-Chasseigne-Imbert, Mou-Swiech. . .

2) Showing that nonlinear Neumann problems can be analyzed via integro-differential methods, e.g. homogenization for nonlinear Neumann problems (Guillen-Schwab 2015).

Ideas of the proof : Incorrect intuition, norm is a max

$$\|x\|_V = \max_{\Lambda \in V^*,\ \|\Lambda\| \leq 1} (|\Lambda(x)|)$$

- $f : \mathbb{R}^d \to \mathbb{R}$, **Lipschitz** (Touching above by cones)

$$f(x) - f(y) \leq C\,|x - y| \implies f(x) = \min_{y \in \mathbb{R}^d} (f(y) + |x - y|)$$

$$\begin{aligned} f(x) &= \min_{y \in \mathbb{R}^d} \max_{\Lambda \in (\mathbb{R}^d)^*, \|\Lambda\| \leq 1} (f(y) + \Lambda(x - y)) \\ &= \min_{y \in \mathbb{R}^d} \max_{p \in \mathbb{R}^d, \|p\| \leq 1} (f(y) + p \cdot (x - y)) \end{aligned}$$

GREAT! $y \mapsto p \cdot y$ is linear with the GCP.

Ideas of the proof : Incorrect intuition, norm is a max

$$\|x\|_V = \max_{\Lambda \in V^*,\ \|\Lambda\| \leq 1} (|\Lambda(x)|)$$

- $I : C^2(\mathbb{R}^d) \to C(\mathbb{R}^d)$ **Lipschitz** (Touching by cones)

$$I(u,x) - I(v,x) \leq C\|u - v\| \implies I(x) = \min_{v \in C^2} (I(v,x) + \|u - v\|)$$

$$I(u,x) = \min_{v \in C^2} \max_{\Lambda \in (C^2)^*, \|\Lambda\| \leq 1} (I(v,x) + \Lambda(u - v))$$
$$= ?????$$

OOOPS! Why can we limit ourselves to only those Λ that also have the GCP?!?!?!

Ideas of the proof : Assume too much – Fréchet diff.

Lemma

If I has the GCP and I is Fréchet differentiable at u, then DI_u also has the GCP.

Suppose that $\phi \leq 0$ everywhere with $\phi(x_0) = 0$.

Then, for any $s \geq 0$

$$u \text{ touches } u + s\phi \text{ from above at } x_0$$

$$\Rightarrow I(u + s\phi, x_0) \leq I(u, x_0) \;\; \forall\, s \geq 0$$

$$\Longrightarrow \; DI_u(\phi, x_0) = \lim_{s \to 0} \frac{I(u + s\phi, x_0) - I(u, x_0)}{s} \leq 0.$$

Ideas of the proof : Assume too much – Fréchet diff.

Assume I is Fréchet differentiable, then

$$\begin{aligned} I(u,x) - I(v,x) &= \int_0^1 (DI)_{tu+(1-t)v}(u-v,x)\, dt \\ &= L(u-v,x), \ \ \forall\, u,v \in C^2 \end{aligned}$$

Given u and v, the linear operator

$$L = \int_0^1 (DI)_{tu+(1-t)v}\, dt$$

lies in the convex hull of the image of $f \to (DI)_f$, i.e.

$$L \in \mathcal{D} := \text{hull}\{L \mid L = (DI)_u, \text{ for some } u \in C^2\}$$

Ideas of the proof : Assume too much – Fréchet diff.

Then, for every $v \in C^2$ define

$$K_v(u,x) := \max_{L \in \mathcal{D}} \{ I(v,x) + L(u-v,x) \}$$

The key fact is that (from the previous calculation)

$$I(u,x) \leq K_v(u,x), \;\; \forall \; u,v \in C^2, \; x \in M$$

But the above is an equality when $v = u$, so

$$I(u,x) = \min_{v \in C^2} \max_{L \in \mathcal{D}} \{ f_{vL}(x) + L(u,x) \}$$

$$\text{where } f_{vL}(x) := I(v,x) - L(v,x) \in C^0.$$

Ideas of the proof : I is only Lipschitz

Operators which are only Lipschitz are very important (e.g. extremal operators)

Problem: If dim$=\infty$, Lipschitz maps are not necessarily Fréchet differentiable in a dense set. So, essentially $\mathcal{D} = \emptyset$.

Strategy:

- Prove analogue theorem for **finite graphs**.
- Approximate manifold M by a sequence of finite graphs.
- Project the operator I to the graph, preserving GCP property.
- Obtain min-max formula in the graph. Pass to the limit.

First, let us prove the finite graph version of the theorem.

Ideas of the proof : finite dimensional case

Let X be a **finite** set, let $C(X)$ denote the space of all real valued functions in X and

$$L : C(X) \to C(X)$$

Lemma

Assume $L : C(X) \to C(X)$ *is* **linear**. *Then* $\exists\ K(x,y) \geq 0$ *defined for* $x \neq y$ *and* $c(x) \leq 0$ *such that*

$$L(u,x) = c(x)u(x) + \sum_{y \in X \setminus \{x\}} (u(y) - u(x))K(x,y)$$

Proof of the lemma

For any $y \in X$, define

$$e_y(x) := \begin{cases} 1 & x = y, \\ 0 & x \neq y. \end{cases}$$

Clearly,

$$u(x) = \sum_{y \in X} u(y) e_y(x) \ \ \forall \ u$$

L is linear, so

$$L(u,x) = \sum_{y \in X} u(y) L(e_y(\cdot), x)$$

This may be conveniently rearranged as

$$L(u,x) = u(x) \sum_{y \in X} L(e_y(\cdot), x) + \sum_{y \in X} (u(y) - u(x)) L(e_y(\cdot), x)$$

Ideas of the proof : finite dimensional case

This can be rewritten as

$$L(u,x) = c(x)u(x) + \sum_{y \in X \setminus \{x\}} (u(y) - u(x))\, K(x,y)$$

where

$$\begin{aligned} K(x,y) &:= L(e_y(\cdot), x), \quad \forall\, x, y \in X, x \neq y, \\ c(x) &:= L(\mathbf{1}, x), \quad \forall\, x \in X. \end{aligned}$$

thus $K(x,y) \geq 0$ and $c(x) \leq 0$.

The Clarke Subdifferential

Recall $I : C(G) \to C(G)$ is Lipschitz.

Given $f \in C(G)$, Clarke "subdifferential" of I at f: $(\mathcal{D}I)_f$

$$(\mathcal{D}I)_f := \text{hull}\left\{ L \mid \ \exists \{f_n\} \text{ s.t. } f_n \to f, \ \ L = \lim_n L_n, \ L_n = (DI)_{f_n} \right\}$$

The total subdifferential

$$\mathcal{D}I := \text{hull}\left(\bigcup_{f \in C(G)} (\mathcal{D}I)_f \right)$$

Important: Rademacher's theorem: $(\mathcal{D}I)_f \neq \emptyset \ \ \forall f \in C(G_n)$.

The Clarke Subdifferential

The following is a useful property of the Clarke subdifferential.

Lemma

Let $I : C(G) \to C(G)$*, Lipschitz.*
Given $f, g \in C(G)$*,* $\exists\ L \in \text{hull}(\mathcal{D}I)$ *such that*

$$I(f) = I(g) + L(f - g)$$

Ideas of the proof : finite dimensional case

We can now conclude just as in the Fréchet case:

$$I(f,x) = \min_{g \in C(G)} K_g(f,x)$$
$$= \min_{g \in C(G)} \max_{L \in \mathrm{hull}(\mathcal{D}I)} \{I(g,x) + L(f-g,x)\}$$

Now, we use the GCP property of I:

$\mathrm{hull}(\mathcal{D}I) =$ convex closure of the limits of $\{L_n\}$, with $L_n = (DI)_{f_n}$, all of which have the GCP!

It follows that every $L \in \mathrm{hull}(\mathcal{D}I)$ has the GCP, and the finite graph theorem is proved.

Ideas of the proof : approximation and back again

Disclaimer! This is a gross simplification

For each $n \in \mathbb{N}$, we have

$$G_n := [-2^n, 2^n] \cap (2^{-n}\mathbb{Z}^d)$$
$$G_n \subset G_{n+1}, \quad \overline{(\lim_{n\to\infty} G_n)} = \mathbb{R}^d$$

Stage	key tool
$I : C^2(\mathbb{R}^d) \to C^{\mathbb{R}^d}$ Lipschitz	Whitney Extension + Restriction
$I_n : C(G_n) \to C(G_n)$	$T_n I(E_n^2 u)$
I_n has min-max	proved above
$\mathcal{D}I_n$	proved above
I min-max	"$\|I_n(u) - I(u)\|_{C(\mathbb{R}^d)} \to 0$" $u \in C_c^3(\mathbb{R}^d)$
"$\mathcal{D}I = \lim_{n\to\infty} \mathcal{D}I_n$"	stability of min-max

What you end up with

Min-Max over discrete operators

$$I_n(u,x) = \min_{v \in C_b^\beta} \max_{L_n \in \mathcal{D}I_n} \left(I_n(v,x) + L_n(u-v,x) \right)$$

where

$$\forall x \in G_n, \quad L_n(u,x) = C^n(x)u(x) + \int_{M \setminus \{x\}} u(y) - u(x) \, \mu_x^n(dy).$$

or

$$L_n(u,x) = \mathrm{Tr}(A^{\delta,n}(x) D^2 u(x)) + B^{\delta,n}(x) \cdot \nabla u(x) + C^n(x)u(x) + \int_{M \setminus \{x\}} u(y) - T_x^{\delta,\beta}(u,y) \, \mu_x^n(dy);$$

WARNING!

The measures μ_x^n from the previous slide are SIGNED MEASURES

The D-to-N

$$\begin{cases} F(U_\phi, X) = 0 & \text{in } \Omega \\ U_\phi = \phi & \text{on } \partial\Omega, \end{cases}$$

Assume F is good enough that weak solutions satisfy **regularity**

$$\phi \in C^{1,\gamma}(\partial\Omega) \implies U_\phi \in C^{1,\gamma}(\overline{\Omega}),$$

and **comparison**

$$U_\phi|_{\partial\Omega} = \phi \le \psi = U_\psi|_{\partial\Omega} \implies U_\phi \le U_\psi \text{ in } \overline{\Omega}.$$

Defines the **D-to-N**

$$\mathcal{I} : C^{1,\gamma}(\partial\Omega) \to C^{\gamma}(\partial\Omega)$$
$$\mathcal{I}(\phi, x) = \partial_n U_\phi(x),$$

The D-to-N

- $F(U,X) = -\text{div}(A\nabla U)(X)$, $A \in C^\gamma$ symmetric, uniformly elliptic
- $F(U,X) = \text{Tr}(A(X)D^2U) + B(X) \cdot \nabla U + C(X)U$, $A \in C^\gamma$ symmetric, uniformly elliptic
- $F(U,x) = F(D^2U, \nabla U, x)$, uniformly elliptic, $|F(P,x) - F(P,y)| \leq C\,|x-y|^\gamma\,(1 + \|P\|)$

The D-to-N

Theorem (Bony-Courrège, $\mathcal{I}$ as above, for both linear cases of F,)

$$\mathcal{I}(u,x) = C(x)u(x) + B(x)\cdot\nabla_\tau u(x) + \int_{\partial\Omega\setminus\{x\}} u(y) - u(x) - \chi_{B_{r_0}(x)}(\nabla_\tau u(x), \nabla_\tau \frac{d(x,y)^2}{2}) d\mu_x^{ab}(y)$$

Theorem (Guillen-Schwab arXiv two days ago!, $\mathcal{I}$ as above, F nonlinear)

$$\mathcal{I}(u,x) = \min_a \max_b \Big\{ \; C^{ab}(x)u(x) + B^{ab}(x)\cdot\nabla_\tau u(x) + \int_{\partial\Omega\setminus\{x\}} u(y) - u(x) - \chi_{B_{r_0}(x)}(\nabla_\tau u(x), \nabla_\tau \frac{d(x,y)^2}{2}) d\mu_x^{ab}(y) \; \Big\}$$

The D-to-N

Theorem (Guillen-Schwab arXiv two days ago!... continued)

Furthermore,

1. *For every* $\gamma > 1$ *we have*

$$\sup_{x \in \partial\Omega} \int_{\partial\Omega \setminus \{x\}} \min\{1, d(x,y)^\gamma\} \, d\mu_x^{ab}(y) < \infty$$

2. *There are constants* r_0, c, *and* C *such that for all* $r \in (0, r_0)$

$$cr^{-1} \leq \int_{B_{2r}(x) \setminus B_r(x)} d\mu_x^{ab}(y) \leq Cr^{-1} \;\; \forall \, x \in \partial\Omega, r > 0.$$

For example... (Schwab-Silvestre 2016) Hölder regularity when:

- $\displaystyle\int_{B_{2r}\setminus B_r} K(h)\,\mathrm{d}h \le (2-\alpha)\Lambda r^{-\alpha}$

- For every $r > 0$, there exists a set A_r such that
  - $A_r \subset B_{2r} \setminus B_r$.
  - A_r is symmetric in the sense that $A_r = -A_r$.
  - $|A_r| \ge \mu |B_{2r} \setminus B_r|$.
  - $K(h) \ge (2-\alpha)\lambda r^{-d-\alpha}$ in A_r.

The D-to-N : reverse the flow!

OLD Flow: integro-differential problem on $\partial\Omega \to$ second order extension $\to$ send info to $\partial\Omega$

NEW Flow: second order problem in $\Omega \to$ reduction to int-diff on $\partial\Omega \to$ sind info back to problem in Ω

Good motivation: Homogenization

The D-to-N

Fully nonlinear equations / nonlinear boundary operators

Interesting questions

A small sample of many **open problems**/research directions

1. For what fully nonlinear operators can we prove that the Lévy measures in the min-max formula fall in the Bass-Levin class?
2. Characterize which nonlocal operators arise as D-to-N maps.
3. What if $M = \mathbb{R}^d$ and I is translation invariant? Can the operators in the min-max be taken to be translation invariant as well?.
4. Extend theorem to singular/unbounded operators such as the infinity Laplace and p-Laplace.
5. Obtain bounds on the Lévy kernels $k(x, y)$ when the diffusion coefficients of L degenerate at ∂D. This is related to questions of unique continuation (currently being investigated by M. Boratko at UMass)

The End

Thanks!