Symmetries of solutions to nonlocal problems

Sven Jarohs

Goethe University Frankfurt, Germany
jarohs@math.uni-frankfurt.de

3rd Conference on Nonlocal Operators and Partial Differential Equations in Będlewo

30. June 2016

Theorem (Gidas, Ni, Nirenberg '79)

Let u be a bounded positive solution of

$$\begin{aligned} -\Delta u = f(u) \quad & \text{in } B \\ u = 0 \quad & \text{on } \partial B, \end{aligned}$$

where f is locally Lipschitz continuous, then u is radially symmetric and strictly decreasing in its radial direction.

Theorem (Gidas, Ni, Nirenberg '79)

Let u be a bounded positive solution of

$$\begin{aligned} -\Delta u &= f(u) && \text{in } B \\ u &= 0 && \text{on } \partial B, \end{aligned}$$

where f is locally Lipschitz continuous, then u is radially symmetric and strictly decreasing in its radial direction.

Question: Is it possible to assume $u \geq 0$ in B instead of $u > 0$ in B?

Answer: No.

$$u : [-\pi, \pi] \to \mathbb{R}, \quad u(x) = 1 - \cos(x)$$

is a bounded nonnegative function which satisfies

$$-\frac{d^2}{dx^2} u = u - 1 \quad \text{in } (-\pi, \pi), \qquad u(\pm\pi) = 0,$$

but $u(0) = 0$. In particular, u is not monotone in $(-\pi, \pi)$.

The fractional Laplacian

For $s=\frac{\alpha}{2}\in(0,1)$ and $u\in C_c^2(\mathbb{R}^N)$ the fractional Laplacian is defined as

$$(-\Delta)^s u:=\mathscr{F}^{-1}(|\cdot|^{2s}\mathscr{F}(u)).$$

Moreover, we have for $x\in\mathbb{R}^N$ the following integral-representation

$$(-\Delta)^s u(x)=c_{N,s}\lim_{\varepsilon\to 0}\int_{\{y\in\mathbb{R}^N\,:\,|x-y|>\varepsilon\}}\frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy,$$

with $c_{N,s}:=s(1-s)\frac{4^s\Gamma(\frac{N}{2}+s)}{\pi^{N/2}\Gamma(2-s)}$.

Weak solutions

Denote

$$\mathscr{H}_0^s(B) = \{u : \mathbb{R}^N \to \mathbb{R} \ : \ u = 0 \quad \text{on } \mathbb{R}^N \setminus B \\ \text{and } \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\, dxdy < \infty\}.$$

In the following, for $g \in L^2(B)$ we will call u a solution of

$$\begin{aligned} (-\Delta)^s u = g \quad & \text{in } B \\ u = 0 \quad & \text{on } \mathbb{R}^N \setminus B, \end{aligned}$$

if $u \in \mathscr{H}_0^s(B)$ satisfies for all $\varphi \in \mathscr{H}_0^s(B)$

$$\frac{c_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\, dxdy = \int_B g(x)\varphi(x)\, dx.$$

Theorem (Birkner, López-Mimbela and Wakolbinger '05)

Let u be a bounded nonnegative solution of

$$\begin{aligned} (-\Delta)^s u &= f(u) && \text{in } B \\ u &= 0 && \text{on } \mathbb{R}^N \setminus B, \end{aligned}$$

where $f : [0,\infty) \to [0,\infty)$ be nondecreasing and not constant, then u is radially symmetric.

Theorem (SJ and T. Weth '14)

Let u be a bounded nonnegative solution of

$$\begin{aligned} (-\Delta)^s u = f(u) \quad & \text{in } B \\ u = 0 \quad & \text{on } \mathbb{R}^N \setminus B, \end{aligned}$$

where f is locally Lipschitz continuous, then u is radially symmetric. Moreover, <u>either</u> $u \equiv 0$ on $\mathbb{R}^N$ <u>or</u> u is strictly decreasing in its radial direction and hence $u > 0$ in B.

Theorem (SJ and T. Weth '14)

Let u be a bounded nonnegative solution of

$$\begin{aligned} (-\Delta)^s u = f(u) \quad & \text{in } B \\ u = 0 \quad & \text{on } \mathbb{R}^N \setminus B, \end{aligned}$$

where f is locally Lipschitz continuous, then u is radially symmetric. Moreover, either $u \equiv 0$ on $\mathbb{R}^N$ or u is strictly decreasing in its radial direction and hence $u > 0$ in B.

Corollary

Any nonnegative bounded solution u of

$$\left(-\frac{d^2}{dx^2}\right)^s u = u - 1 \quad \text{in } (-\pi,\pi), \qquad u \equiv 0 \text{ on } \mathbb{R} \setminus (-\pi,\pi)$$

is even and strictly decreasing on $(0,\pi)$. Hence $u > 0$ in $(-\pi,\pi)$.

General version

Theorem (SJ and T. Weth '14)

Let $\Omega \subset \mathbb{R}^N$ be an open bounded Lipschitz set, which is symmetric and convex in x_1 and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be locally Lipschitz in u (uniformly in x) such that f is symmetric in x_1 and

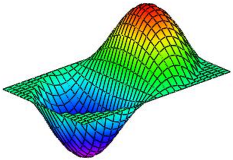
$$f(x_1, x', u) \geq f(x_2, x', u) \quad \text{for } u \in \mathbb{R},\ (x_1, x'), (x_2, x') \in \Omega,\ |x_1| \leq |x_2|.$$

Then every nonnegative bounded solution u of

$$\begin{aligned} (-\Delta)^s u &= f(x,u) && \text{in } \Omega \\ u &= 0 && \text{on } \mathbb{R}^N \setminus \Omega \end{aligned}$$

is symmetric in x_1. Moreover, either $u \equiv 0$ on $\mathbb{R}^N$ or u is strictly decreasing in $|x_1|$ and hence $u > 0$ in Ω.

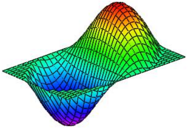
Foliated Schwarz symmetry

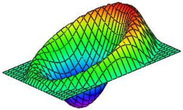


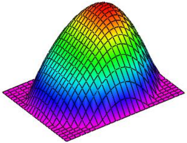
Let $D \subset \mathbb{R}^N$ be a radial set.

A function $u : D \to \mathbb{R}$ is called foliated Schwarz symmetric in D if there is $p \in S^{N-1}$ such that

- *u is axially symmetric w.r.t. $\mathbb{R} \cdot p$ and*
- *u is nonincreasing in the polar angle $\theta = \arccos\left(\frac{x}{|x|} \cdot p\right)$.*







Lemma (F. Brock '03)

Let $D \subset \mathbb{R}^N$ be a radial set, $u : D \to \mathbb{R}$ continuous. Then u is foliated Schwarz symmetric w.r.t. p if and only if for every half space $H \subset \mathbb{R}^N$ with $0 \in \partial H$ and $p \in H$ we have

$$u \geq u \circ Q_H \quad \text{in } H.$$

Here $Q_H : \mathbb{R}^N \to \mathbb{R}^N$ denotes the reflection at ∂H.

Theorem (SJ '16)

Let $D \subset \mathbb{R}^N$ be a radial bounded set and u a bounded solution of

$$\begin{aligned} (-\Delta)^s u &= f(|x|, u) && \text{in } D \\ u &= 0 && \text{on } \mathbb{R}^N \setminus D, \end{aligned}$$

where f is locally Lipschitz continuous.
If there is a half space $H \subset \mathbb{R}^N$ with $0 \in \partial H$ and such that $u \geq u \circ Q_H$ in H, $u \not\equiv u \circ Q_H$, then u is foliated Schwarz symmetric.

Theorem (SJ '16)

Let $D \subset \mathbb{R}^N$ be a radial bounded set and u a bounded solution of

$$\begin{aligned} (-\Delta)^s u = f(|x|, u) \quad & \text{in } D \\ u = 0 \quad & \text{on } \mathbb{R}^N \setminus D, \end{aligned}$$

where f is locally Lipschitz continuous.
If there is a half space $H \subset \mathbb{R}^N$ with $0 \in \partial H$ and such that $u \geq u \circ Q_H$ in H, $u \not\equiv u \circ Q_H$, then u is foliated Schwarz symmetric.

Remark

Also holds for unbounded radial sets, if u satisfies $\lim_{|x| \to \infty} u(x) = 0$ and there is $\delta > 0$ such that $\frac{f(r,u)}{u} \leq 0$ for $u \in [-\delta, \delta]$

The previous results hold if $r \mapsto r^{-N-2s}$, $r > 0$ is replaced with a function $J : (0, \infty) \to [0, \infty)$ which satisfies

1. *J is (strictly) decreasing.*
2. $\int_{B_1(0)} |z|^2 J(|z|)\ dz + \int_{\mathbb{R}^N \setminus B_1(0)} J(|z|)\ dz < \infty$;
3. $\int_{\mathbb{R}^N} J(|z|)\ dz = \infty$.

The previous results hold if $r \mapsto r^{-N-2s}$, $r > 0$ is replaced with a function $J : (0, \infty) \to [0, \infty)$ which satisfies

1. *J is (strictly) decreasing.*
2. $\int_{B_1(0)} |z|^2 J(|z|)\ dz + \int_{\mathbb{R}^N \setminus B_1(0)} J(|z|)\ dz < \infty$;
3. $\int_{\mathbb{R}^N} J(|z|)\ dz = \infty$.

Example

Possible choice $r \mapsto r^{-N} 1_{(0,1)}(r)$ or $r \mapsto -r^{-N} \ln(r) 1_{(0,1)}(r)$

Corollary (SJ '16)

Let $D \subset \mathbb{R}^N$ *be a radial set and* $q \in [2, \frac{2N}{N-2s}]$*. Then every continuous bounded minimizer of* $K : \mathscr{H}_0^s(D) \to \mathbb{R}$*,*

$$K[u] = \frac{c_{N,s}}{4} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} \, dxdy - \int_D F(|x|, u(x)) \, dx$$

which satisfies $\|u\|_{L^q(D)} = 1$ *is foliated Schwarz symmetric.*

Here $F(r,u) = \int_0^u f(r,\tau)\, d\tau$*, where* $f : [0,\infty) \times D \to \mathbb{R}$ *is locally Lipschitz and such that there are constants* $a, b > 0$ *with*

$$|f(r,u)| \le a|u| + b|u|^{q-1} \quad \text{for all } r \ge 0,\ u \in \mathbb{R}.$$

Thank you for your attention.

References:

- T. Weth and S. J. (2014), *Asymptotic Symmetry for Parabolic Equations involving the Fractional Laplacian*
  Discrete Contin. Dyn. Syst. 34.6, 2581–2615.
- T. Weth and S. J. (2016),*Symmetry via antisymmetric maximum principles in nonlocal problems of variable order*
  Ann. Mat. Pura Appl. (4), 195.1, 273–291.
- S. J. (2015), *Symmetry via maximum principles for nonlocal nonlinear boundary value problems*, thesis.
- S. J. (2016), *Symmetry of solutions to nonlocal nonlinear boundary value problems in radial sets*,
  NoDEA 23.3, 1–22.