Heat kernels of regularly/slowly varying convolution semigroups

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Based on joint project with W. Cygan, M. Ryznar and B. Trojan

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Let N be rotational invariant Lévy measure ($\int 1 \wedge |z|^2 N(\mathrm{d}z) < \infty$) and $\sigma \geq 0$. Define, for $u \in C_0^2(\mathbb{R}^d)$,

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We define

$$\psi(\xi) = \int_{\mathbb{R}^d} (1 - \cos(\xi z))N(\mathrm{d}z).$$

Then we have

$$\mathcal{F}(\mathscr{L}u)(\xi) = -\psi(\xi)\mathcal{F}(u)(\xi).$$

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Examples: $s \in (0, 1]$

- $-(-\Delta)^s \longrightarrow \quad \psi(\xi) = |\xi|^{2s}$
- $-(-\Delta + I)^s + I, \longrightarrow \quad \psi(\xi) = (|\xi|^2 + 1)^s - 1$
- $-(\log(1 - \Delta))^s, \longrightarrow \quad \psi(\xi) = \left(\log(1 + |\xi|^2)\right)^s$

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We examine the fundamental solution of the heat equation

$$\mathscr{L} p(t,x) = \partial_t p(t,x), \quad t > 0,\, x \in \mathbb{R}^d.$$

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Our goal: asymptotics and estimates for $p(t,x)$.

Strong ratio limit

Example:
For Δ^s we have $\psi(\xi) = |\xi|^{2s}$ and

$$p(t,x) = t^{-d/(2s)} p(1, xt^{-1/(2s)}) = t^{-d/(2s)} p\left(1, (t\psi(1/|x|))^{-1/(2s)}\right).$$

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Hence

$$\frac{p(t,x)}{p(t,0)} \longrightarrow 1, \quad \text{if } t\psi(1/|x|) \to \infty.$$

Strong ratio limit

Proposition (Cygan-TG-Trojan)

Assume that there exist α, $c > 0$ *such that*

$$\psi(\lambda\xi) \geq c\lambda^{\alpha}\psi(\xi), \quad \lambda \geq 1,\, \xi \in \mathbb{R}^d.$$

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We have

$$p(t,0) = (2\pi)^{-d} \int e^{-t\psi(\xi)} \mathrm{d}\xi.$$

Regular variation

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- $f \in \mathcal{R}_\alpha^\infty$ if

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- $f(r) = r^\alpha \log^\gamma(1 + r^\beta) \in \mathcal{R}_{\alpha+\gamma\beta}^0 \cap \mathcal{R}_\alpha^\infty$, $\alpha, \gamma \in \mathbb{R}$, $\beta \geq 0$.

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- *If $\psi \in \mathcal{R}_\alpha^\infty$ for some $\alpha > 0$ then*

$$\frac{p(t,0)}{\left[\psi^{-1}\left(\frac{1}{t}\right)\right]^d} \xrightarrow{t\to 0^+} C(d,\alpha).$$

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Example:
For $\psi(\xi) = |\xi|^\alpha \log^\alpha(1 + |\xi|^\alpha)$, $\alpha \in \left(0, \frac{2}{3}\right]$, we have

$$t^{d/\alpha} \log^d(t) p(t,0) \xrightarrow{t\to 0^+} C(d,\alpha).$$

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Equivalently $t\psi\left(\frac{1}{|x|}\right) \to 0^+$. Hence

$$\frac{|x|^d}{t\psi\left(\frac{1}{|x|}\right)} p(t,x) = \left(|x|t^{-1/(2s)}\right)^{d+2s} p\left(1, xt^{-1/(2s)}\right) \xrightarrow{t\psi\left(\frac{1}{|x|}\right)\to 0} \mathcal{A}(d,s).$$

Small $t\psi$

From now on we assume that $N(\mathrm{d}z) = \nu(|z|)\mathrm{d}z$, where

$$\nu : (0, \infty) \mapsto [0, \infty) \quad \text{and } \nu - \text{non-increasing.}$$

That means

$$\mathscr{L}u(x) = P.V. \int (u(x+z) - u(x))\nu(|z|)\mathrm{d}z + \sigma\Delta.$$

Important property $p(t, \cdot)$ is radial and radially non-increasing.

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Example:
Let $\varphi(\lambda) = \sigma\lambda + \int_0^\infty (1 - e^{-\lambda s})\mu(ds)$ (a Bernstein function). Then $-\varphi(-\Delta)$ has the above form with

$$\nu(r) = \int_0^\infty (4\pi s)^{-d/2} e^{-\frac{r^2}{4s}} \mu(ds).$$

Main results

Theorem (Cygan-TG-Trojan)

- *If* $\psi \in \mathcal{R}_\alpha^\infty$, *for some* $\alpha \in (0,2)$ *then*

$$\frac{|x|^d p(t,x)}{t\psi\left(\frac{1}{|x|}\right)} \xrightarrow[x\to 0]{t\psi\left(\frac{1}{|x|}\right)\to 0} \mathcal{A}(d,\alpha).$$

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Lemma

Let $\nu(r) = r^{-d} g(r^{-1})$ *and* $\alpha \in (0, 2)$.

- *If* $g \in \mathcal{R}_{\alpha}^{\infty}$, *then* $\psi \in \mathcal{R}_{\alpha}^{\infty}$.
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Example:
Let $\nu(r) = r^{-d-\alpha} \log^\beta(1 + e^\beta + r)$, for $\alpha \in (0,2)$, $\beta \geq 0$. Then

$$\frac{p(t,x)}{t|x|^{-d-\alpha} \log^\beta |x|} \xrightarrow[|x| \to \infty]{t|x|^{-\alpha} \log^\beta |x| \to 0} 1$$

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Similar result holds also for $\mathcal{R}_\alpha^0$.

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Similar result holds also for $\mathcal{R}^{0}_{\alpha}$.

Question: what with $\alpha = 0$ and $\alpha = 2$?

For $\alpha = 2$ see paper by Mimica and Kim (preprint 2016).

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Hence

$$p(t,x) \approx t\left(\psi\left(\frac{e}{|x|}\right) - \psi\left(\frac{1}{|x|}\right)\right)|x|^{-d} e^{-t\psi\left(\frac{1}{|x|}\right)}.$$

Slowly varying symbols, $\psi \in \mathcal{R}_0$

Definition

- $f \in \Pi_\ell^\infty$, for $\ell \in \mathcal{R}_0^\infty$ if

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For instance $\psi(\xi) = \log(1 + |\xi|)$. We have

$$\psi(\lambda r) - \psi(r) = \log\left(\frac{1+\lambda r}{1+r}\right) \xrightarrow{r\to\infty} \log \lambda.$$

Hence $\psi \in \Pi_\ell^\infty$ with $\ell \equiv 1$.

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Similar result holds also for Π_ℓ^0.

Asymptotics

Theorem (TG-Ryznar-Trojan)

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- $$\frac{|x|^d \nu(|x|)}{\ell\left(\frac{1}{|x|}\right)} \xrightarrow{x\to 0} C > 0.$$

In particular $C = \frac{\Gamma(d/2)}{2\pi^{d/2}}$.

Similar result holds also for Π_ℓ^0.

Recall that for $\psi(\xi) = \log(1 + |\xi|)$ we have

$$p(t,x) \approx \frac{t|x|^t}{|x|^d}, \quad \text{for } t,\ |x| \text{ small.}$$

By the above theorem

$$p(t,x) \sim C\frac{t}{|x|^d}, \quad x \to 0 \text{ and } |x|^t \to 1.$$

Observe that $p(t,0) = \infty$ and $e^{-t\psi} \notin L^1$.

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Remark: If $\ell \to \infty$ at infinity then $p(t,0) < \infty$.

Estimates

Let $\psi \in \Pi_\ell^\infty$ for $\ell \in \mathcal{R}_0^\infty$ and ℓ be bounded. We have

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And

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Proposition (TG-Ryznar-Trojan)

There exist c, C such that for any non-increasing ν

$$p(t,x) \geq Ct\nu(|x|)e^{-ct\psi\left(\frac{1}{|x|}\right)}, \quad t > 0,\ x \in \mathbb{R}^d.$$

Lemma (TG-Ryznar-Trojan)

Let $\psi \in \Pi_\ell^\infty$ for $\ell \in \mathcal{R}_0^\infty$ and ℓ be bounded. Then there is $C > 0$ such that,

$$p(t,x) \leq t|x|^{-d}\ell\left(\frac{1}{|x|}\right) e^{-t\psi\left(\frac{1}{|x|}\right)}, \quad t, |x| \text{ small.}$$

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Hence

Theorem (TG-Ryznar-Trojan)

Let $\psi \in \Pi_\ell^\infty$ for $\ell \in \mathcal{R}_0^\infty$ and ℓ be bounded.
Then

$$p(t,x) \approx t|x|^{-d}\ell\left(\frac{1}{|x|}\right)e^{-t\psi\left(\frac{1}{|x|}\right)}, \quad t, |x| \text{ small}.$$

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$[-d, 0)$	bdd	$t\nu(\lvert x\rvert)$
2	???	$t\left(\psi(\lvert x\rvert^{-1} - (2\lvert x\rvert)^{-1}\psi'(\lvert x\rvert^{-1})\right)\lvert x\rvert^{-d}$

Bounded ψ

We have

$$p(t,x) = e^{-tN(\mathbb{R}^d)} \sum_{k=1}^{\infty} \frac{t^k}{k!} N^{*k}(x)$$

$$= e^{-tN(\mathbb{R}^d)} t\nu(|x|) \left(1 + t \sum_{n=0}^{\infty} \frac{t^n}{(n+2)!} \frac{N^{*(n+2)}(x)}{\nu(|x|)} \right).$$

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Hence, for t and $|x|$ small we have

$$p(t,x) \sim t\nu(|x|).$$

Example

Let $\psi(\xi) = \log(1 + |\xi|^\alpha)$, $\alpha \in (0,2)$.

Then, for $t < 3d/\alpha$,

$$p(t,x) \approx \begin{cases} t|x|^{-d-\alpha}, & |x| \geq 1, \\ t \min\left\{ \log(2|x|^{-\alpha}), (t - d/\alpha)^{-1} \right\}, & |x| < 1,\ t > d/\alpha, \\ t(\log(2|x|^{-\alpha}) + |x|^{\alpha t - d}), & |x| < 1,\ t \in (0, d/\alpha]. \end{cases}$$

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And for $t \geq 3d/\alpha$,

$$p(t,x) \approx \min\left\{ t^{-d/\alpha}, t|x|^{-d-\alpha} \right\}.$$

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Green function

Let

$$G(x) = \int_0^\infty p(t,x)dt.$$

Riesz potential, $\psi(\xi) = |\xi|^{2s}$,

$$G(x) = c|x|^{2s-d} = \frac{c}{|x|^d \psi\left(|x|^{-1}\right)}.$$

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Theorem

Suppose $d \geq 6$. Then

$$\lim_{x\to 0} \frac{G(x)}{|x|^{-d}\psi(|x|^{-1})^{-1}} = c > 0, \tag{1}$$

if and only if $\psi \in \mathcal{R}_\alpha^\infty$, for some $\alpha > 0$. In particular, (1) implies that

$$c = 2^{-\alpha}\pi^{d/2}\frac{\Gamma\left((d-\alpha)/2\right)}{\Gamma\left(\alpha/2\right)}.$$

Let

$$U_t(r) = \mathbb{P}\big(0 < |X_t| \leq \sqrt{r}\big) = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^{\sqrt{r}} u^{d-1} p(t,u)\,\mathrm{d}u.$$

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We observe that by the Tonelli's theorem

$$\begin{aligned}
\lambda \mathcal{L} U_t(\lambda) &= \int_{\mathbb{R}^d} e^{-\lambda|x|^2} p(t,x)\,\mathrm{d}x. \\
&= (4\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-t\psi(\xi\sqrt{\lambda})} e^{-|\xi|^2/4}\,\mathrm{d}\xi - \mathbb{P}\big(|X_t| = 0\big) \\
&= \frac{2^{1-d}}{\Gamma(d/2)} \int_0^\infty e^{-t\psi(r\sqrt{\lambda})} e^{-r^2/4} r^{d-1}\,\mathrm{d}r - \mathbb{P}\big(|X_t| = 0\big)
\end{aligned}$$

Thank you.