Estimates for solutions of fractal Burgers equation

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Fractional Laplacian

- Let $d \in \mathbb{N}$ and $\alpha \in (0,2)$. We define $p \colon (0,\infty) \times \mathbb{R}^d \to \mathbb{R}$

$$\int_{\mathbb{R}^d} p(t,z) e^{iz\cdot\xi} dz = e^{-t|\xi|^\alpha}, \quad \xi \in \mathbb{R}^d, \quad t > 0,$$

- Stable semigroup

$$P_t f(x) = \int_{\mathbb{R}^d} p(t,x,y) f(y) dy,$$

where $p(t,x,y) := p(t, y-x)$.

- Fractional Laplacian - generator of stable semigroup

$$\Delta^{\alpha/2} f(x) = \lim_{t\to 0^+} \frac{P_t f(x) - f(x)}{t}$$
$$= \lim_{\varepsilon\to 0^+} c \int_{|y|>\varepsilon} \frac{f(x+y) - f(x)}{|y|^{d+\alpha}} dy\,, \quad f \in C_c^\infty(\mathbb{R}^d)\,.$$

Properties of $p(t,x)$

- scaling property

$$p(t,x,y) = t^{-d/\alpha} p(1, t^{-1/\alpha} x)$$

- estimates

$$c_1 \frac{t}{(|x| + t^{1/\alpha})^{d+\alpha}} \le p(t,x) \le c_2 \frac{t}{(|x| + t^{1/\alpha})^{d+\alpha}}$$

- gradient estimates

$$|\nabla_x p(t,x)| \le c \frac{p(t,x)}{|x| + t^{1/\alpha}}$$

heat equation with $\Delta^{\alpha/2}$

Consider the (pseudo)differential equation in $\mathbb{R}^d$ ($\alpha \in (0,2)$).

$$\begin{cases} \partial_t u = \Delta^{\alpha/2} u, \\ u(0,x) = \delta_0(x), \end{cases}$$

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Now consider

$$\begin{cases} \partial_t u = \Delta^{\alpha/2} u, \\ u(0,x) = u_0(x) \in L^1(\mathbb{R}^d), \end{cases}$$

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Solution:

$$u(t,x) = P_t u_0(x) = \int_{\mathbb{R}^d} p(t,x,y) u_0(y) dy,$$

Burgers equation for fractional Laplacian

Let $\alpha \in (1, 2)$. Consider a nonlinear boundary value problem

$$\begin{cases} \partial_t u(t,x) = \Delta^{\alpha/2} u(t,x) + b \cdot \nabla_x (u^{q+1}(t,x)) \\ u(0,x) = u_0(x), \end{cases}$$

where

- $b \in \mathbb{R}^d$ – constant vector
- $u_0 \geq 0$, $u_0 \in L^1(\mathbb{R}^d)$
- $q = \frac{\alpha - 1}{d}$ - critical exponent

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- For $d = 1$, $\alpha = 2$ and $b = 1$ we get classical Burgers equation

$$\partial_t u(t,x) = \partial_x^2 u(t,x) + \partial_x (u^2(t,x))$$

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- Interesting special case $u_0 = M\delta_0, \quad M > 0$
  $u_M(t,x)$ – source solution

motivation

- Biler, Funaki, Woyczyński, (1998)

  $\Longrightarrow$ existence of solutions

- Biler, Karch, Woyczyński, (1999)

  $\Longrightarrow u_t = u_{xx} + \Delta^{\alpha/2} u + u u_x = 0$

- Biler, Karch, Woyczyński, (2001)

  $\Longrightarrow$ existence of solutions, L^p estimates, regularity

- Brandolese, Karch, (2008) $\Longrightarrow$ estimates $u_M(t,x) \le Cp(t,x)$ for $u_0 = M\delta_0$ and small M

- Many other papers devoted to this type of equation (Alibaud, Andreianov, Droniou, Imbert, Kiselev, Nazarov, Shterenberg, Wang, . . .)

Properties of solutions

$$\begin{cases} \partial_t u(t,x) = \Delta^{\alpha/2} u(t,x) + b \cdot \nabla_x (u^{q+1}(t,x)) \\ u(0,x) = u_0(x), \end{cases}$$

Biler, Karch and Woyczyński, (2001):

- $u(t,x) > 0$ for $t > 0$ and $x \in \mathbb{R}^d$
- scaling property for $u_0 = M\delta_0$, $q = (\alpha - 1)/d$

$$u_M(t,x) = t^{-d/\alpha} u_M(1, t^{-1/\alpha} x)$$

  (no scaling for $u_0 \in L^1$)
- L^p estimates

$$\|u(t,\cdot)\|_p \leq cMt^{-\frac{d}{\alpha}(1-\frac{1}{p})}$$

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Biler, Karch and Woyczyński, (2001):

- for $q = \frac{\alpha-1}{d}$, $u_0 \in L^1(\mathbb{R}^d)$ and $\|u(t,\cdot)\|_1 = M$,

$$t^{d(1-1/p)/\alpha} \|u(t,\cdot) - u_M(t,\cdot)\|_p \to 0, \qquad \text{as } t \to \infty.$$

- for $q > \frac{\alpha-1}{d}$ and $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$

$$t^{d(1-1/p)/\alpha} \|u(t,\cdot) - P_t u_0(\cdot)\|_p \to 0, \qquad \text{as } t \to \infty.$$

main results

Theorem (T.J., G. Serafin, *Nonlinear Analysis*, 2016)

Let $u_0 = M\delta_0$. *For arbitrary* $M > 0$ *there is a constant* $C > 0$ *such that*

$$C^{-1}p(t,x) \le u_M(t,x) \le Cp(t,x), \qquad t > 0,\ x \in \mathbb{R}^d$$

Furthermore

$$|\nabla_x u_M(t,x)| \le C\frac{p(t,x)}{|x| + t^{1/\alpha}}$$

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Theorem (T.J., G. Serafin, *preprint*, 2015)

Let $u_0 \in L^1(\mathbb{R}^d)$. *There is a constant* $C > 0$ *such that*

$$C^{-1}P_t u_0(x) \leq u(t,x) \leq CP_t u_0(x), \qquad t > 0,\ x \in \mathbb{R}^d$$

the case $u_0 = M\delta_0$ – main ideas of the proof

- Consider a boundary value problem

$$\begin{cases} \partial_t u(t,x) = \Delta^{\alpha/2} u(t,x) + Bu(t,x) \\ u(0,x) = u_0(x), \quad M > 0 \end{cases}$$

where $Bu = b \cdot \nabla_x u^{q+1}$

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- $$\begin{cases} \partial_t \widehat{u}(t,\xi) = -|\xi|^\alpha \widehat{u}(t,\xi) + \widehat{Bu}(t,\xi) \\ \widehat{u}(0,\xi) = \widehat{u}_0(\xi) \end{cases}$$

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$$\Longrightarrow \qquad \widehat{u}(t,\xi) = \widehat{u}_0(\xi) e^{-t|\xi|^\alpha} + \int_0^t e^{-(t-s)|\xi|^\alpha} \widehat{Bu}(s,\xi)\, ds$$

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(Duhamel formula - Perturbation formula)

$$u(t,x) = P_t u_0(x) + \int_0^t \int_{\mathbb{R}^d} p(t-s,x,z) Bu(s,z) \, dz \, ds$$

the case $u_0 = M\delta_0$ – main ideas of the proof

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$\Longrightarrow \quad$ we prove only $u_M(x) \approx p(1,x)$ for large x.

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\begin{aligned}
&\int_0^1 \int_{\mathbb{R}^d} p(1-s,x,z) b \cdot \nabla_z [u_M(s,z)]^{q+1}\, dz\, ds \\
&- \int_0^1 \int_{\mathbb{R}^d} b \cdot \nabla_z p(1-s,x,z) [u_M(s,z)]^{q+1}\, dz\, ds \\
&= -\alpha \int_0^1 \int_{\mathbb{R}^d} b \cdot \nabla_w p(1-r^\alpha, x, rw) [u_M(1,w)]^{q+1}\, dw\, dr
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$$= -\alpha \int_0^1 \int_{\mathbb{R}^d} b \cdot \nabla_w p(1-r^\alpha, x, rw) [u_M(1,w)]^{q+1}\, dw\, dr$$

- Hence,

$$u_M(x) = Mp(1,x) - \alpha \int_{\mathbb{R}^d} \overbrace{\int_0^1 b \cdot \nabla_w p(1-r^\alpha, x, rw)\, dr}^{\leq H(x,w)} [u_M(w)]^{q+1}\, dw$$

the case $u_0 = M\delta_0$ – main ideas of the proof

- Put $$H(x,w) = \int_0^1 (1-r^\alpha)^{-1/\alpha} p(1-r^\alpha, x, rw) dr$$

$$u_M(x) \leq Mp(1,x) + c \int_{\mathbb{R}^d} H(x,w)[u_M(w)]^{q+1}\, dw\, dr$$

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- For $R > 0$, we denote

$$h_R(x) = \int_{B(0,R)} H(x,w)[u_M(1,w)]^{1+q}dw,$$
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Lemma

If $|x| \to \infty$, then $u_M(x) \to 0$

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- for sufficiently large $|x|$, we have

$$H_R(x) \le Cp(1,x) + Ch_R(x) \qquad \text{and} \qquad h_R(x) \le Cp(1,x)$$

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  u^* depends on t, but plays similar role as $u_M(x)$

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$$P_t^* u_0(x) = t^{d/\alpha} P_t u_0(t^{1/\alpha} x) = t^{d/\alpha} \int_{\mathbb{R}^d} p(t, t^{1/\alpha} x, y) u_0(y) dy$$

  (P_t^* is not a semigroup)

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- we get

$$u^*(t,x) \leq P_t^* u_0(x) + C \int_0^1 \int_{\mathbb{R}^d} |\nabla_w p(1 - r^\alpha, x, rw)| [u^*(r^\alpha t, w)]^{q+1} dw dr$$

properties of $u^*(t, x)$

- $\|u^*(t,\cdot)\|_p \leq C\|u_0\|_1 \qquad$ for $p \in [1,\infty]$
- $\lim_{t\to 0} \|u^*(t,\cdot)\|_\infty = 0$
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Corollary

For $t \le t_0$ or $|x| \ge R$, we have $u^*(t,x) \approx P_t^* u_0(x)$

$(t,x) \in (t_0,\infty) \times B(0,R)$

- $P_t^* u_0(x) \approx 1$

  $\Longrightarrow$ we need to show $c_1 \leq u^*(t,x) \leq c_2$
- upper bound follows from estimates $\|u^*(t,\cdot)\|_p \leq C\|u_0(\cdot)\|_1$
- lower bound:

  Let $u_\varepsilon(t,x)$ be solution to

  $$\begin{cases} \partial_t u(t,x) = \Delta^{\alpha/2} u(t,x) + b \cdot \nabla_x (u^{q+1}(t,x)) \\ u(0,x) = \varepsilon u_0(x), \end{cases}$$

  - $u_\varepsilon(t,x) > c > 0$ for small ε
  - $u(t,x) \geq u_\varepsilon(t,x)$ for $0 < \varepsilon < 1$

the case $q > \frac{\alpha-1}{d}$

- $$\begin{cases} \partial_t u(t,x) = \Delta^{\alpha/2} u(t,x) + b \cdot \nabla_x (u^{q+1}(t,x)) \\ u(0,x) = u_0(x) \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d), \end{cases}$$

Theorem (T.J., G. Serafin, *preprint*, 2015)

Let $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and $q > \frac{\alpha-1}{d}$. There is a constant $C > 0$ such that

$$C^{-1} P_t u_0(x) \le u(t,x) \le C P_t u_0(x), \qquad t > 0,\ x \in \mathbb{R}^d$$

Furthermore,

$$\lim_{|x|\to\infty} \sup_{t>0} \left\| \frac{u(t,x)}{P_t u_0(x)} - 1 \right\|_\infty = 0$$