Martin kernels for Markov processes with jumps

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Introduction

Boundary Harnack inequality and existence of limits

The boundary Harnack inequality is a statement about positive harmonic functions in an open set D, which are equal to zero on a part of the boundary. It states that if D is regular enough (for example, a Lipschitz domain), x_0 is a boundary point of D, f and g are positive and harmonic in D, and both f and g converge to 0 on $\partial D \cap B(x_0, R)$, then for every $r \in (0, R)$ the ratio f/g has bounded *relative oscillation* in $D \cap B(x_0, r)$:

$$\sup_{x \in D \cap B(x_0,r)} \frac{f(x)}{g(x)} \le c_{BHI} \inf_{x \in D \cap B(x_0,r)} \frac{f(x)}{g(x)}.$$

The estimate above turns out to be self-improving as $r \to 0^+$, in the sense that the constant c_{BHI} converges to 1 as $r \to 0^+$. Equivalently, the boundary limit

$$\lim_{\substack{x \to x_0 \\ x \in D}} \frac{f(x)}{g(x)}$$

exsists.

Martin representation and Martin boundary

Martin representation asserts that there is a one-to-one correspondence between positive harmonic functions f in D and positive measures μ on D_m – the *Martin boundary of D*. The two objects are linked by the formula

$$f(x) = \int_{\partial D_m} M_D(x, z) \mu(dz),$$

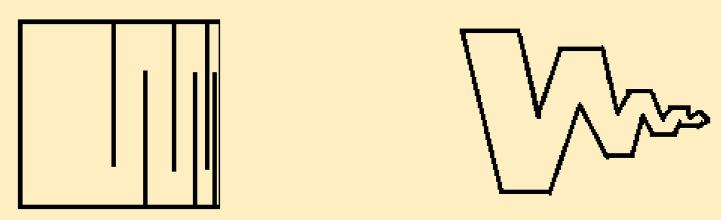
where the *Martin kernel* is defined as the boundary limit of the ratio of *Green functions*:

$$M_D(x, z) = \lim_{\substack{y \to z \\ x \in D}} \frac{G_D(x, y)}{G_D(\tilde{x}, y)}.$$

Here $\tilde{x} \in D$ is an arbitrarily fixed reference point.

The Martin compactification of D is the smallest compact extension of D that allows for continuous extensions of ratios of Green functions $M_D(x, y) = G_D(x, y)/G_D(x_0, y)$. Martin boundary $\partial_M D$ is just the boundary of D in this extension, and the Martin kernel $M_D(x, z)$ at $z \in \partial_M D$ is the corresponding limit of $M_D(x, y)$ as $y \to z$. The minimal Martin boundary is a subset of the Martin boundary that consists of *minimal* harmonic functions. Given the existence of limits on the boundary, $D_m = \partial D$, but in other cases we may observe some interesting phenomena:

pic. 1 topological boundary of the set (left) and its "bigger" Martin boundary (right)



pic. 2 topological boundary of the set (left) and its "smaller" Martin boundary (right)

Generalisations of BHI for local operators

BHI can be considered not only for Laplace operator, but also for other local operators. History of BHI for local operators is very rich, however this field is outside our interest.

Generalisations of BHI for nonlocal operators

The history of the boundary Harnack inequality for nonlocal operators starts with the article by K. Bogdan in 1997, where he proved the result for the fractional Laplace operator $-(-\Delta)^{\alpha/2}$ and Lipschitz domains. This was extended later to more general sets and operators:

For fractional Laplacian:

- Song–Wu: κ-fat sets (disconnected analogues of NTA domains)
- Bogdan–Kulczycki–Kwaśnicki: arbitrary open sets

More general opreators:

- Bogdan–Burdzy–Chen, Guan: for censored stable processes (regional fractional Laplacian)
- Kim–Song–Vondraček: for subordinate Brownian motions (sufficiently regular translation-invariant operators with isotropic kernels) and beyond
- Chen–Kim–Song–Vondraček: for $-\Delta + (-\Delta)^{\alpha/2}$ (mixture of local and non-local)

Recently a rather general result for nonlocal operators was proved by K. Bogdan, T. Kumagai and M. Kwaśnicki, and this is our starting point in the study of boundary limits.

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Our results

Notation and definitions

- Let m be a reference measure on some metric space $\mathscr{X}$.
- We consider the bilinear form:

$$\mathscr{E}(f, g) = \int_{\mathscr{X}} \int_{\mathscr{X}} (f(x) - f(y))(g(x) - g(y)) \nu(x, y) m(dx) m(dy),$$

where $\nu(x, y)$ is a positive kernel.

- The above form corresponds to an operator $-L$, in the sense that $\mathscr{E}(f, g) = \langle -Lf, g \rangle$ for f in the domain of L.
- Informally, L is a singular integral operator

$$Lf(x) = \mathrm{P.V.}\int_{\mathfrak{X}} (f(y) - f(x)) \nu(x, y) m(dy);$$

however, it is rather difficult to define the above principal value intergal in the general context.

- We impose a number of conditions on ν, which, in particular, assert that L generates a Markovian semigroup of operators $P_t = e^{tL}$, and that the Green function $G_D(x, y)$ for $-L$ exists in an arbitrary bounded open set D.
- Note: the Green function is the kernel of the Green operator $G_D f(x) = \int_D G_D(x, y) f(y) m(dy)$, which satisfies $\mathscr{E}(G_D f, g) = \langle f, g \rangle$ for all g in the domain of $\mathscr{E}$ equal to zero outside D (that is, informally, $LG_D f = -f$).

Assumptions

All the assumptions in our paper are written in language of stochastic processes. Here however we will try to rewrite them in language of nonlocal operators

(A) P_t is both Feller (maps C_0 into C_0) and strong Feller (maps L^∞ int C_b)

(B) There is a linear subspace $\mathcal{D}$ of $\mathcal{D}(L)$ satisfying the following condition. If K is compact, D is open, and $K \subseteq D \subseteq \mathfrak{X}$, then there is $f \in \mathcal{D}$ such that $f(x) = 1$ for $x \in K$, $f(x) = 0$ for $x \in \mathfrak{X} \setminus D$, $0 \le f(x) \le 1$ for $x \in \mathfrak{X}$, and the boundary of the set $\{x : f(x) > 0\}$ has measure m zero.

(C) If $x_0 \in \mathfrak{X}$, $0 < r < R < R_0$, $x \in B(x_0, r)$ and $y \in \mathfrak{X} \setminus B(x_0, R)$, then

$$c^{-1} \nu(x_0, y) \le \nu(x, y) \le c \nu(x_0, y),$$

for some $c > 0$.

(D) If $x_0 \in \mathfrak{X}$, $0 < r < s < R < R_0$ and $B = B(x_0, R)$, then

$$\sup_{x \in B(x_0,r)} \sup_{y \in \mathfrak{X} \setminus B(x_0,s)} G_B(x, y) < \infty. \tag{1}$$

The above assumptions are needed for the BHI proven by K. Bogdan, T. Kumagai, M. Kwaśnicki in case $\nu(x, y) = \nu(y, x)$, that is, L is symmetric. If L is not symmetric then assumptions A - D need to be satisfied also by the dual (adjoint) operator L^*, corresponding to the kernel $\nu^*(x, y) = \nu(y, x)$. In order to prove existence of the limits one needs to assume also that constants appearing in assumptions B - D and constant c_{BHI} from K. Bogdan, T. Kumagai, M. Kwaśnicki theorem are scale invariant.

Main theorem

Let $D \subseteq \mathfrak{X}$ be open, $x_0 \in \partial D$ and $R > 0$. Suppose that operator L satisfies the above assumptions; Suppose furthermore that nonnegative functions f and g are regular harmonic functions in $D \cap B(x_0, R)$ and are equal to zero in $B(x_0, R) \setminus D$. Then either one of f and g is zero everywhere in D, or the finite, positive boundary limit of $f(x)/g(x)$ exists as $x \to x_0$, $x \in D$. Furthermore,

$$\lim_{\substack{x \to x_0 \\ x \in D}} \frac{f(x)}{g(x)} = \lim_{r \to 0^+} \frac{\int_{\mathfrak{X} \setminus B(0,r)} \nu(x_0, y) f(y) m(dy)}{\int_{\mathfrak{X} \setminus B(0,r)} \nu(x_0, y) g(y) m(dy)}.$$

Remark

This theorem immediately implies the existence of the Martin kernel $M_D(x, z)$, defined as the boundary limit of ratios of Green functions. We conclude that for *bounded* open sets D, the Martin boundary of D coincides with the topological boundary.

Example 1

Let $\nu(x, y) = |y - x|^{d+\alpha} \varphi\left(\frac{y-x}{|y-x|}\right)$ with φ continuous and positive on the unit sphere in $\mathbb{R}^d$ be the kernel of L. Such operator satisfies assumptions A - D. The formula for Lf can be given explicitly:

- If $\alpha < 1$ then $Lf(x) = c\mathrm{P.V.}\int_{\mathbb{R}^d} (f(y) - f(x)) \nu(x, y) dy$
- If $\alpha = 1$ then $Lf(x) = b\nabla f(x) + c * \mathrm{P.V.}\int_{\mathbb{R}^d} (f(y) - f(x)) \nu(x, y) dy$
- If $\alpha > 1$ then $Lf(x) = c\mathrm{P.V.}\int_{\mathbb{R}^d} (f(y) - f(x) - (y - x)\nabla f(x)) \nu(x, y) dy$

Other examples

Operator L with kernel ν given by the formula

$$\nu(x, y) = \varphi(x, y) |x - y|^{-d-\alpha},$$

where φ is symmetric (that is, $\varphi(x, y) = \varphi(y, x)$), bounded by positive constants, smooth, and has bounded partial derivatives of all orders.

Reference

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