Reduced measures for semilinear elliptic equations with Dirichlet operator

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Let E be a locally compact separable metric space and m a Radon measure on E such that $\operatorname{supp}[m] = E$. Let us consider the following problem:

$$-Au = f(x,u) + \mu, \tag{1}$$

- $(A, D(A))$ is selfadjoint non-positive definite Dirichlet operator on $L^2(E;m)$,
- $f : E \times \mathbb{R} \to \mathbb{R}$ is continuous and nonincreasing with respect to u (no assumptions on the growth!),
- μ is a Borel measure on E, $f(x,u) = 0$, $u \le 0$,
- for every $y \in \mathbb{R}$, $x \mapsto f(x,y) \in qL^1(E;m)$ (note that $L^1(E;m) \subset qL^1(E;m)$).

Example

$A = \Delta^\alpha$, $\alpha \in (0,1]$, $N \ge 2$, $\beta > 0$, $p \ge 1$

$$f(x,u) = -\frac{1}{|x|^\beta}|u|^{p-1}u, \quad f(x,u) = \frac{1}{|x|^\beta}(1 - e^u).$$

Let $(\mathcal{E}, D[\mathcal{E}])$ be a Dirichlet form associated with A and Cap be a capacity which for open relatively compact sets $U \subset E$ is given by

$$\mathrm{Cap}(U) = \inf\{\mathcal{E}(u,u) : u \in D[\mathcal{E}],\ u \geq \mathbf{1}_U,\ m\text{-p.w.}\}.$$

Every Borel measure μ admits the decomposition

$$\mu = \mu_d + \mu_c, \quad \mu_d \ll Cap, \quad \mu_c \perp Cap.$$

Theorem (Brezis, Strauss (1973), Konishi (1972))

If $\mu \in L^1(E;m)$ (i.e. $\mu_c \equiv 0$, $\mu_d \in L^1(E;m)$) and for some $\alpha > 0$, $\alpha\|v\|_{L^1} \leq \|Av\|_{L^1}$, $v \in D(A)$, then there exists a unique solution of (1).

Theorem (Klimsiak, Rozkosz (2013))

If μ is bounded and $\mu_c \equiv 0$, then there exists a unique solution of (1).

Example (Bénilan, Brezis (1975))

Let $U \subset \mathbb{R}^N$ be an open bounded set, $0 \in U$, $N \geq 3$ and $p \geq \frac{N}{N-2}$. There is no solution of the following equation

$$-\Delta u = f(u) + \mu \quad \textit{on} \quad U, \quad u_{|\partial U} = 0,$$

where $f(u) = -|u|^{p-1}u$, $\mu = \delta_0$.

In that work as a solution understood a function u on U such that $u \in L^p(U; dx)$ (equivalently $f(u) \in L^1(U; dx)$)

$$-\int_U u\Delta\eta = \int_U f(u)\eta + \int_U \eta d\mu,$$

for every function

$$\eta \in C_0^2(\overline{U}) \equiv \{\eta \in C^2(\overline{U});\ \eta = 0 \text{ na } \partial U\}.$$

It is known now, that for every $\mu \perp \mathrm{Cap}$, there exists a continuous nonincreasing function f such that there is no solution for (1).

It is known that for every $n \geq 1$ there exists a solution of

$$-\Delta u_n = f(u_n) \vee (-n) + \mu \quad \text{on} \quad U, \quad u_{n|\partial U} = 0,$$

By comparison theorem sequence $\{u_n\}$ is nonincreasing, so it is well defined

$$u^* = \lim_{n \to \infty} u_n.$$

Theorem (Brezis, Marcus, Ponce, 2007)

We have $u^* \in L^1(E; m)$, $f(u^*) \in L^1(E; m)$ and there exists a Borel measure μ^* such that

$$-\Delta u^* = f(u^*) + \mu^* \quad \textit{on} \quad U, \quad u^*_{|\partial U} = 0.$$

Moreover μ^* is the biggest Borel measure less then μ for which there exists a solution of (1).

Let $\{T_t,\ t \geq 0\}$ be a Markov semigroup generated by $(-A, D(A))$. For $f \equiv 0$

$$-Au = \mu, \text{ so } \quad u = (-A)^{-1}\mu.$$

If $\mu = g \in L^{1,+}(E; m)$, then we put

$$Rg = (-A)^{-1}g \equiv \lim_{t\to\infty} \int_0^t T_s g\, ds.$$

Since R is nonnegative, there exists a kernel $\{R(x, dy),\ x \in E\}$ such that

$$Rg(x) = \int_E g(y)R(x, dy).$$

Observe that if $\mu = \delta_x$, then u is the Green function, which is a density for measure $R(x, dy)$, so it is natural to assume that

$$R(x, dy) = r(x, y)m(dy).$$

for some Borel measurable function $r : E \times E \to \mathbb{R}^+$. Now we may define

$$R\mu(x) = \int_E r(x, y)\, \mu(dy).$$

Definition

We say that u is a solution of (1) iff

$$u(x) = \int_E f(y, u(y)) r(x, y)\, dy + \int_E r(x, y) \mu(dy), \quad q.e.$$

Let us consider the class of measures

$$\mathbb{M} = \{\mu\text{-Borel measure on } E;\, R|\mu| < \infty,\ m\text{-a.e.}\},$$

It is known that $\mathcal{M}_b \subset \mathbb{M}$. Let $(\{X_t,\ t \geq 0\}, \{P_x,\ x \in E\}, \zeta)$ be a Markov process (Hunt process) associated with $\{T_t,\ t \geq 0\}$, i.e.

$$T_t f(x) = E_x f(X_t) \equiv \int f(X_t)\, dP_x.$$

Definition

We say that a function u is a solution of (1) iff

(a) $f(\cdot,u)\cdot m\in\mathbb{M}$ *and there exists a local MAF M such that*

$$u(X_t)=u(X_0)-\int_0^t f(X_r,u(X_r))\,dr - \int_0^t dA_r^{\mu_d}+\int_0^t dM_r,\quad t\geq 0,\quad P_x\text{-a.s.}$$

for q.e. $x\in E$,

(b) *for every polar* $N\subset E$ *and sequence of stopping times* $\{\tau_k\}$ *such that* $\tau_k\nearrow T\geq\zeta$, $E_x\sup_{t\leq\tau_k}|u(X_t)|<\infty$ $x\in E\setminus N$, $k\geq 1$ *we have*

$$E_x u(X_{\tau_k})\to R\mu_c(x),\quad x\in E\setminus N,$$

Regularity of solutions

Theorem

Let u be a solution of (1) with $\mu \in \mathcal{M}_b$ and $f(\cdot,0) \in L^1(E;m)$. Then for every $k \geq 0$, $T_k(u) \in D_e[\mathcal{E}]$ and

$$\mathcal{E}(T_k(u), T_k(u)) \leq 2k(\|f(\cdot,0)\|_{L^1} + \|\mu\|_{TV}),$$

where

$$T_k(u) = (u \wedge k) \vee (-k).$$

Remark

(i) In particular $T_k(u) \in D[\mathcal{E}]$, when $m(E) < \infty$, because $D[\mathcal{E}] = D_e[\mathcal{E}] \cap L^2(E;m)$.

(ii) $T_k(u) \in D[\mathcal{E}]$ if there exists $c > 0$ such that $c(u,u) \leq \mathcal{E}(u,u)$ for every $u \in D[\mathcal{E}]$, since then $D_e[\mathcal{E}] = D[\mathcal{E}]$.

Definition

We say that $\mu \in \mathbb{M}$ *is a good measure if there exists a solution of (1). The set of good measures we denote by* $\mathcal{G}$.

Theorem

Niech $f_n = f \vee (-n)$ *oraz* u_n *będzie rozwiązaniem*

$$-Au_n = f_n(x, u_n) + \mu.$$

Then $u_n \searrow u^*$, *where* u^* *is a maximal subsolution (1). Moreover* $\mu^* \equiv -Au^* - f(x, u^*)$ *ia a maximal good measure less then* μ, $\mu^* = \mu_d + \nu$ *with* $\nu \perp \mathrm{Cap}$ *such that* $\nu \leq \mu_c$.

Now we may define mapping $^* : \mathbb{M} \to \mathcal{G}$ putting μ^* derived from the above theorem. It is clear that for every $\mu \in \mathcal{G}$ we have $\mu^* = \mu$. In particular for every measure $\mu \in \mathbb{M}$ absolutely continuous with respect to Cap we have $\mu = \mu^*$.

For every strictly positive excessive function ρ (i.e. $T_t\rho \le \rho$) we put

$$\mathcal{M}_\rho = \{\mu \in \mathbb{M};\ \|\mu\|_\rho \equiv \int \rho\, d|\mu| < \infty\}.$$

Observe that $\mathbb{M} = \bigcup_\rho \mathcal{M}_\rho$.

Proposition

(i) *$\mathcal{G}$ is convex,*

(ii) $\|\mu - \mu^*\|_\rho = \min_{\nu \in \mathcal{G}_\rho} \|\mu - \nu\|_\rho$ *for every* $\mu \in \mathcal{M}_\rho$.

Let us consider the following class of mesures
$\mathcal{A}_\rho(f) = \{\mu \in \mathbb{M} : f(\cdot, R\mu) \in L^1(E; \rho \cdot m)\}$.

Proposition

$\mathcal{A}_\rho(f) + L^1(E; \rho \cdot m) = \mathcal{G}_\rho$.

(A) For every $\theta \in [0,1)$, $c \ge 0$ there exist $\alpha(c,\theta)$, $\beta(c,\theta) \ge 0$ s.t.

$$|f(x, \theta u + c)| \le \alpha(c,\theta)|f(x,u)| + \beta(c,\theta), \quad x \in E,\ u \in \mathbb{R}.$$

Theorem

Let $\rho \in L^1(E;m)$. If (A) holds, then $\overline{\mathcal{A}_\rho(f)} = \mathcal{G}_\rho$, where closure is taken in the space $(\mathcal{M}_\rho, \|\cdot\|_\rho)$.

Theorem

Let $\mu \in \mathbb{M}$ and u be a solution of (1). If $u \geq 0$, then $\mu_c \geq 0$.

Theorem

Let u be a solution of (1). Then $Au^+ \in \mathbb{M}$ and

$$\mathbf{1}_{\{u>0\}}(Au)_d \leq (Au^+)_d,$$

$$(Au)_c^+ = (Au^+)_c.$$

Examples

Let $D \subset \mathbb{R}^N$ be open and bounded, $N \geq 2$. Let us consider for $p > 1$ the following equation

$$-\Delta u = -|u|^{p-1}u + \mu \quad \text{na } D, \quad u = 0 \quad \text{on } \partial D. \tag{2}$$

Let us also consider the following capacity

$$\mathrm{Cap}_{W^{2,p}}(K) = \inf\{\|\eta\|^p_{W^{2,p}}; \varphi \in C_c^\infty(D) \text{ and } \varphi \geq 1 \text{ on } K\}.$$

Theorem (Barras, Pierre (1984))

There exists a solution of (2) if and only if $\mu \ll Cap_{W^{2,p'}}$.

Corollary

$\mathcal{G}$ *is a linear space and* $\mu^* = \mu^+_{Cap_{W^{2,p'}}} - \mu^-$.

If $p < \frac{N}{N-2}$, then $\mathrm{Cap}_{W^{2,p'}}(B) = 0$ if and only if $B = \emptyset$.

Przykłady

Let $D \subset \mathbb{R}^N$ be an open bounded set of class C^2, $N \geq 2$. Let us consider for $p > 1$ the following equation

$$-\Delta^\alpha u = -|u|^{p-1}u + \mu \quad \text{na } D, \quad u = 0 \quad \text{na } \mathbb{R}^N \setminus D. \qquad (3)$$

Let us consider the following capacity

$$\mathrm{Cap}_{W^{2\alpha,p}}(K) = \inf\{\|\eta\|^p_{W^{2\alpha,p}}; \varphi \in C_c^\infty(D) \text{ oraz } \varphi \geq 1 \text{ na } K\}.$$

Theorem (L. Véron (2013))

There exists a solution of (3) if and only if $\mu \ll \mathrm{Cap}_{W^{2\alpha,p'}}$.

Corollary

$\mathcal{G}$ *is a linear space and* $\mu^* = \mu^+_{\mathrm{Cap}_{W^{2\alpha,p'}}} - \mu^-$.

Let $D \subset \mathbb{R}^N$ be open and bounded. Let us consider the following equation

$$-\Delta u = 1 - e^u + \mu \quad \text{na } D, \qquad u = 0 \quad \text{on } \partial D. \tag{4}$$

Theorem (Vázquez (1983))

Assume that $N = 2$. There exists a solution of (4) if and only if $\mu(\{x\}) \leq 4\pi$ for every $x \in D$.

Corollary

$\mathcal{G}$ is not a linear space and under decomposition $\mu = \mu_1 + \sum_{n\geq 1} \alpha_n \delta_{x_n}$ on nonatomic part μ_1 and purely atomic part $\sum_{n\geq 1} \alpha_n \delta_{x_n}$ we have $\mu^* = \mu_1 + \sum_{n\geq 1} \min\{4\pi, \alpha_n\} \delta_{x_n}$.

Theorem (Bartolucci, Leoni, Orsina, Ponce (2005))

Assume that $N \geq 3$. If $\mu \in \mathcal{M}_1$ and $\mu \leq 4\pi \mathcal{H}^{N-2}$, then there exists a solution of (4).

Let us consider the following equation

$$-Au = -|u|^{p-1}u + \mu. \tag{5}$$

Let us consider the following capacity

$$\mathrm{Cap}_{A,p}(B) = \inf\{\int_E |A\eta|^p\, m(dx);\ \eta \in D(A_p),\ \eta \ge 1 \text{ on } B\}$$

Theorem

There exists a solution of (5) if and only if $\mu \ll Cap_{A,p'}$.

Corollary

If $\mu \in \mathbb{M}$, then $\mu^* = \mu^+_{Cap_{A,p'}} - \mu^-$.

References

- Brezis, H., Marcus, M., Ponce, A.C.: Nonlinear elliptic equations with measures revisited. In: Mathematical Aspects of Nonlinear Dispersive Equations (J. Bourgain, C. Kenig, S. Klainerman, eds.), Annals of Mathematics Studies, **163**, Princeton University Press, Princeton, NJ, 55–110 (2007)
- Chen, H., Véron, L.: Semilinear fractional elliptic equations involving measures. J. Differential Equations **257**, 1457–1486 (2014)
- Klimsiak, T., Rozkosz, A.: Dirichlet forms and semilinear elliptic equations with measure data. J. Funct. Anal. **265**, 890–925 (2013)
- Klimsiak, T.: Reduced measures for semilinear elliptic equations involving Dirichlet operators. (2016) Calc. Var. Partial Differ. Equ. (to apear)