# *Supersolutions in Fractional Nonlinear Problems*[1]

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Based on joint works with J. Korvenpää & E. Lindgren & G. Palatucci

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[1] A video of an extended version of the talk available at the Fields institute webpage (http://www.fields.utoronto.ca/video-archive/event/2022)

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# Scope

The topic of the talk concerns nonlocal and nonlinear equations modelled by the fractional $p$-Laplacian in $\mathbb{R}^n$ given by

$$(-\Delta)_p^s u(x) := \mathrm{p.v.} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{n+sp}} \, dy,$$

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where $s \in (0,1)$ and $p \in (1,\infty)$.

- When $p = 2$, it is the fractional Laplacian (up to a normalizing constant).
- For $p \geq 2$ the, operator $(-\Delta)_p^s$ is usually called fractional $p$-Laplacian.
- Arises naturally from minimization of the $W^{s,p}$-seminorm. Some aspects of the talk are relevant already in the case $p = 2$.

# *Scope*

Our goal is to understand Dirichlet boundary value problems in a bounded open set $\Omega$ of the type

$$\begin{cases} (-\Delta)_p^s u = 0 & \text{in } \Omega \subseteq \mathbb{R}^n \\ u = g & \text{a.e. on } \mathbb{R}^n \setminus \Omega\,, \end{cases}$$

where boundary values $g$ can be "rough" and the domain $\Omega$ "irregular".

## *Scope: Perron method*

For given boundary datum $g$, the goal is to find some classes of functions, say $\mathcal{U}_g$ (or $\mathcal{L}_g$), with correct boundary values, and which are lower (or upper) directed. This means that if $u, v \in \mathcal{U}_g$ (or $\mathcal{L}_g$), then $\min(u, v) \in \mathcal{U}_g$ (or $\max(u, v) \in \mathcal{L}_g$).

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$$\overline{H}_g(x) := \inf_{u \in \mathcal{U}_g} u(x) \qquad \left(\text{or } \underline{H}_g(x) := \sup_{u \in \mathcal{L}_g} u(x)\right)$$

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- Solvability: Both $\overline{H}_g$ and $\underline{H}_g$ are solutions in $\Omega$
- Resolutivity: $\overline{H}_g = \underline{H}_g$
- Wiener criterion: Give a necessary and sufficient condition for the geometry of the boundary at a point $z \in \partial\Omega$ such that if $g$ is "continuous at $z$", then so are $\overline{H}_g$ and $\underline{H}_g$.

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It turns out that this is still doable in many nonlinear situations via comparison principles.

Question: Can one do this in the case of the operator $(-\Delta)_p^s$?

# *Different notions of supersolutions*

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1. *Weak supersolutions*. Satisfy the weak formulation against smooth test functions together with fractional integrability.
2. $(s,p)$-*superharmonic functions*. These are defined via comparison against weak solutions.
3. $(s,p)$-*viscosity supersolutions*. The notion of viscosity solutions is based on the pointwise evaluation of the principal value appearing in the definition of $(-\Delta)_p^s$:

$$(-\Delta)_p^s u(x) := \mathrm{p.v.} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}}\, dy.$$

# *Fractional Sobolev space*

We first define the natural energy spaces for the problem. The factional Sobolev space $W^{s,p}$ is defined via Gagliardo-seminorm

$$[u]_{W^{s,p}(\Omega)} := \left( \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}}\, dx\, dy \right)^{1/p}$$

as

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) \, : \, [u]_{W^{s,p}(\Omega)} < \infty \right\}$$

The local version $W^{s,p}_{\mathrm{loc}}(\Omega)$ is defined in an obvious way.

## *Tail space*

The second space controls the behavior of tails:

$$L^{p-1}_{sp}(\mathbb{R}^n) := \left\{ u \in L^{p-1}_{\mathrm{loc}}(\mathbb{R}^n) \,:\, \int_{R^n} \frac{|u(x)|^{p-1}}{(1+|x|)^{n+sp}}\, dx < \infty \right\}.$$

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In particular, $u \in L^{p-1}_{sp}(\mathbb{R}^n)$ implies that in the definition of $(-\Delta)^p_s u$ the nonlocal contributions are finite:
If $|u(x)| < \infty$ and $u \in L^{p-1}_{sp}(\mathbb{R}^n)$, then

$$\left| \int_{\mathbb{R}^n \setminus B_r(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}}\, dy \right| < \infty$$

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Moreover, a very natural quantity

$$\mathrm{Tail}(u; z, r) := \left( r^{sp} \int_{\mathbb{R}^n \setminus B_r(z)} \frac{|u(x)|^{p-1}}{|x-z|^{n+sp}}\, dx \right)^{\frac{1}{p-1}}$$

appearing in the theory frequently is finite for all $z \in \mathbb{R}^n$ and $r > 0$ provided that $u \in L^{p-1}_{sp}(\mathbb{R}^n)$.

## *Weak solutions: Definition*

Let us begin with the definition

*Definition*

We say that $u$ is a weak supersolution to $(-\Delta)_p^s u = 0$ in $\Omega$ if $u \in W^{s,p}_{\mathrm{loc}}(\Omega) \cap L^{p-1}_{sp}(\mathbb{R}^n)$ satisfies

$$\langle (-\Delta)_p^s u, \phi \rangle := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\phi(x) - \phi(y))}{|x-y|^{n+sp}} \, dx \, dy \geq 0$$

whenever $\phi$ belongs to $C_0^\infty(\Omega)$ and is nonnegative. Similarly, $u$ is a weak subsolution if $-u$ is a weak supersolution, and $u$ is a weak solution if it is both sub- and supersolution.

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whenever $\phi$ belongs to $C_0^\infty(\Omega)$ and is nonnegative. Similarly, $u$ is a weak subsolution if $-u$ is a weak supersolution, and $u$ is a weak solution if it is both sub- and supersolution.

The definition still makes sense with measurable coefficients, i.e., if the kernel $|x - y|^{-n-sp}$ is replaced with $K(x, y) = a(x, y)|x - y|^{-n-sp}$, where $a$ is symmetric, measurable and satisfies for example $\Lambda^{-1} \leq a(x, y) \leq \Lambda$ for a.e. $x, y$ with a constant $\Lambda \geq 1$.

## Initial motivation for the problem: Minimization

Weak solutions naturally appear in minimization problems. Defining, for $g \in W^{s,p}(\mathbb{R}^n)$,

$$\mathcal{K}_g(\Omega) := \{u \in W^{s,p}(\mathbb{R}^n) \,:\, u = g \;\text{ a.e. on }\; \mathbb{R}^n \setminus \Omega\} ,$$

then the minimizer of

$$\min_{u \in \mathcal{K}_g(\Omega)} [u]_{W^{s,p}(\mathbb{R}^n)}$$

is a weak solution in $\Omega$, with boundary values $g$, by the first variation.

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There are various ways to generalize this. For instance, letting $\Omega'$ be such that $\Omega \Subset \Omega'$ and defining, for $g \in W^{s,p}(\Omega') \cap L^{p-1}_{sp}(\mathbb{R}^n)$,

$$\mathcal{K}_g(\Omega, \Omega') := \left\{u \in W^{s,p}(\Omega') \cap L^{p-1}_{sp}(\mathbb{R}^n) \,:\, u = g \ \text{ a.e. on } \ \mathbb{R}^n \setminus \Omega\right\} \,,$$

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it is possible to obtain existence and uniqueness of minimizers, which, in turn, are weak solutions in $\Omega$.
Problem: What can be said if $g$ does not belong to $W^{s,p}(\Omega')$ for any $\Omega' \Supset \Omega$?

## *Weak solutions: Some properties*

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- Many other recent results
- Higher regularity still open

# *Weak supersolutions*

Weak supersolutions play an important role in the theory.

Some basic properties of supersolutions are the following (Korvenpää-K-Palatucci, preprint):

- A weak supersolution has a lower semicontinuous (l. s. c.) representative
- Class of uniformly globally bounded weak supersolutions is closed w.r.t. pointwise convergence
- Minimum of two weak supersolutions is a weak supersolution as well
- A nonnegative supersolution $u$ in $B_R(z)$ satisfies

$$\left( \fint_{B_r} u^q(x)\, dx \right)^{1/q} \lesssim \operatorname*{ess\,inf}_{B_r} u + \left( \frac{r}{R} \right)^{\frac{sp}{p-1}} \mathrm{Tail}(\max(-u, 0); z, R)$$

whenever $r \in (0, R/2)$. Here $q \in \left(0, \frac{n(p-1)}{n-ps}\right)$ for $n > ps$ and $q \in (0, \infty)$ for $n \leq ps$.

Recall: $\mathrm{Tail}(f; z, r) := \left( r^{sp} \int_{\mathbb{R}^n \setminus B_r(z)} \frac{|f(x)|^{p-1}}{|x-z|^{n+sp}}\, dx \right)^{\frac{1}{p-1}}$

# $(s,p)$-*superharmonic functions*

*Definition*

We say that a function $u : \mathbb{R}^n \to [-\infty, \infty]$ is an $(s,p)$-*superharmonic function* in an open set $\Omega$ if it satisfies the following four assumptions:

(i) $u < +\infty$ almost everywhere and $u > -\infty$ everywhere in $\Omega$,

(ii) $u$ is lower semicontinuous (l. s. c.) in $\Omega$,

(iii) $u$ satisfies the comparison in $\Omega$ against solutions bounded from above; that is, if $D \Subset \Omega$ is an open set and $v \in C(\overline{D})$ is a weak solution in $D$ such that $\max\{v, 0\} \in L^\infty(\mathbb{R}^n)$ and $u \geq v$ on $\partial D$ and almost everywhere on $\mathbb{R}^n \setminus D$, then $u \geq v$ in $D$,

(iv) $u_-$ belongs to $L^{p-1}_{sp}(\mathbb{R}^n)$.

A function $u$ is $(s,p)$-*subharmonic* in $\Omega$ if $-u$ is $(s,p)$-superharmonic in $\Omega$, and if both $u$ and $-u$ are $(s,p)$-superharmonic, we say that $u$ is $(s,p)$-*harmonic*.

# *Properties of $(s, p)$-superharmonic functions*

*Theorem (Korvenpää-K-Palatucci, preprint)*

*Suppose that $u$ is $(s,p)$-superharmonic in an open set $\Omega$. Then it has the following properties:*

*(i)* **Pointwise behavior**.

$$u(x) = \liminf_{y \to x} u(y) = \operatorname{ess} \liminf_{y \to x} u(y) \quad \textit{for every } x \in \Omega.$$

*(ii)* **Summability**. *For*

$$\overline{t} := \begin{cases} \frac{(p-1)n}{n-sp}, & 1 < p < \frac{n}{s}, \\ +\infty, & p \geq \frac{n}{s}, \end{cases} \qquad \overline{q} := \min\left\{ \frac{n(p-1)}{n-s}, p \right\},$$

*and $h \in (0, s)$, $t \in (0, \overline{t})$ and $q \in (0, \overline{q})$,*
$u \in W^{h,q}_{\mathrm{loc}}(\Omega) \cap L^t_{\mathrm{loc}}(\Omega) \cap L^{p-1}_{sp}(\mathbb{R}^n)$.

*(iii)* **Connection to weak supersolutions**. *If $u$ is locally bounded in $\Omega$ or $u \in W^{s,p}_{\mathrm{loc}}(\Omega)$, then it is a weak supersolution in $\Omega$.*

# *Perron solutions*

We may now define the upper (and lower) classes for the Perron solution

*Definition* (**Perron solutions**)

Let $\Omega$ be an open set. Assume that $g \in L^{p-1}_{sp}(\mathbb{R}^n)$. The upper class $\mathcal{U}_g$ of $g$ consists of all functions $u$ such that

*(i)* $u$ is $(s,p)$-superharmonic in $\Omega$,

*(ii)* $u$ is bounded from below in $\Omega$,

*(iii)* $\liminf_{\Omega \ni y \to x} u(y) \geq \operatorname{ess\,lim\,sup}_{\mathbb{R}^n \setminus \Omega \ni y \to x} g(y)$ for all $x \in \partial\Omega$,

*(iv)* $u = g$ almost everywhere in $\mathbb{R}^n \setminus \Omega$.

The lower class is $\mathcal{L}_g := \{u : -u \in \mathcal{U}_{-g}\}$. Define

$$\overline{H}_g := \inf \{u : u \in \mathcal{U}_g\} \quad \text{and} \quad \underline{H}_g := \sup \{u : u \in \mathcal{L}_g\} .$$

The definition (and some work) guarantees that $\overline{H}_g \geq \underline{H}_g$.

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The definition (and some work) guarantees that $\overline{H}_g \geq \underline{H}_g$.

*Theorem (Korvenpää-K-Palatucci, preprint)*

*The Perron solution $\overline{H}_g$ ($\underline{H}_g$) can be either identically $+\infty$ in $\Omega$, identically $-\infty$ in $\Omega$, or $(s,p)$-harmonic in $\Omega$.*

# A few examples

The function

$$u(x) = c_{n,s} \left(1 - |x|^2\right)^s \int_{\mathbb{R}^n \setminus B_1(0)} g(y) \left(|y|^2 - 1\right)^{-s} |x - y|^{-n} \, dy, \quad x \in B_1(0),$$

solves $(-\Delta)^s u = 0$ in $B_1(0)$ with $u = g$ on $\mathbb{R}^n \setminus B_1(0)$.

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*Example*

Taking the function $g(x) = \left||x|^2-1\right|^{s-1}$, $g \in L^1_{2s}(\mathbb{R}^n)$, as boundary values in the Poisson formula above, the integral does not converge. This example suggests that in this case $\overline{H}_g \equiv \underline{H}_g \equiv +\infty$ in $B_1(0)$. The example also tells that one can not expect bounded solutions for all $g \in L^1_{2s}(\mathbb{R}^n)$.

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*Example*

Let us consider the previous example with $g$ reflected to the negative side in the half space, i. e., $g(x)=\text{sign}(x_n)\left||x|^2-1\right|^{s-1}$. Then the "solution" via Poisson formula, for $x\in B_1$, is $u(x)=\text{sign}(x_n)\cdot\infty$. which is suggesting that we should now have $\overline{H}_g\equiv+\infty$ and $\underline{H}_g\equiv-\infty$ in $B_1(0)$: failure of resolutivity in $L^1_{2s}(\mathbb{R}^n)$?

# *"Viscosity" solutions*

Let us finally comment another possible class for upper solutions, namely viscosity supersolutions.

*Definition*

We say that a function $u : \mathbb{R}^n \to [-\infty, \infty]$ is an $(s,p)$-*viscosity supersolution* in $\Omega$ if it satisfies the following four assumptions.

*(i)* $u < +\infty$ almost everywhere in $\mathbb{R}^n$, and $u > -\infty$ everywhere in $\Omega$.

*(ii)* $u$ is lower semicontinuous in $\Omega$.

*(iii)* If $\phi \in C^2(B_r(x_0))$[3] for some $B_r(x_0) \subseteq \Omega$ is such that $\phi(x_0) = u(x_0)$ and $\phi \le u$ in $B_r(x_0)$, then $(-\Delta)^s_p \phi_r(x_0) \ge 0$, where

$$\phi_r(x) = \begin{cases} \phi(x), & x \in B_r(x_0), \\ u(x), & x \in \mathbb{R}^n \setminus B_r(x_0). \end{cases}$$

*(iv)* $u_-$ belongs to $L^{p-1}_{sp}(\mathbb{R}^n)$.

[3] Replacing $C^2$ with Dini-$C^{\frac{sp}{p-1}}$ for $p > \frac{2}{2-s}$ leads to the same class. The case $1 < p \le \frac{2}{2-s}$ needs an extra assumption on the critical set of $\phi$

# *Equivalence*

It turns out that the classes of $(s,p)$-superharmonic functions and $(s,p)$-viscosity solutions are the same.

*Theorem (Korvenpää-K-Lindgren)*

*A function $u$ is $(s,p)$-superharmonic in $\Omega$ if and only if it is an $(s,p)$-viscosity supersolution in $\Omega$.*

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As an immediate corollary we get:

*Theorem (Korvenpää-K-Lindgren)*

*A function $u$ is a continuous weak solution to $(-\Delta)_p^s u = 0$ in $\Omega$ if and only if it is an $(s, p)$-viscosity solution in $\Omega$.*

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- Let $\phi$ be s.t. $u \geq \phi$ in $B_r(z)$ and $u(z) = \phi(z)$. Assume on contrary that $(-\Delta)_p^s \phi_r(z) < 0$.

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- $(-\Delta)_p^s(\phi_r + \delta\eta)(\cdot) < 0$ for small enough $\delta > 0$ and $\eta$ a cut-off function in the neighborhood of $z$ by the counter assumption (this is where the most of the computations happen)

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- Show that

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- "Integrate by parts" to show that $\phi_r + \delta\eta$ is a weak subsolution
- Use the comparison principle for weak super- and subsolutions to reach a contradiction

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- As in the first part, one can show that $v$ is an $(s,p)$-viscosity subsolution
- Conclude the proof by proving a comparison principle for viscosity solutions (MUCH easier than in the case of local viscosity solutions)

Thank you for your attention!