Supersolutions in Fractional Nonlinear Problems

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Based on joint works with J. Korvenpää & E. Lindgren & G. Palatucci

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1A video of an extended version of the talk available at the Fields institute webpage (http://www.fields.utoronto.ca/video-archive/event/2022)
2sites.google.com/site/tuomokuusimath/
Scope

The topic of the talk concerns nonlocal and nonlinear equations modelled by the fractional $p$-Laplacian in $\mathbb{R}^n$ given by

\[( -\Delta )_p^s u(x) := \text{p.v.} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{n+sp}} \, dy, \]

where $s \in (0,1)$ and $p \in (1, \infty)$. 

• When $p = 2$, it is the fractional Laplacian (up to a normalizing constant).
• For $p \geq 2$ the operator $(-\Delta)^s_p$ is usually called fractional $p$-Laplacian.
• Arises naturally from minimization of the $W_{s,p}^s$-seminorm. Some aspects of the talk are relevant already in the case $p = 2$. 
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- Arises naturally from minimization of the $W^{s,p}$-seminorm. Some aspects of the talk are relevant already in the case $p = 2$. 
Our goal is to understand Dirichlet boundary value problems in a bounded open set $\Omega$ of the type

$$\begin{aligned}
\left\{ (-\Delta)_p^s u &= 0 \quad \text{in } \Omega \subseteq \mathbb{R}^n \\
u &= g \quad \text{a.e. on } \mathbb{R}^n \setminus \Omega,
\right.
\end{aligned}$$

where boundary values $g$ can be “rough” and the domain $\Omega$ “irregular”.
**Scope: Perron method**

For given boundary datum $g$, the goal is to find some classes of functions, say $\mathcal{U}_g$ (or $\mathcal{L}_g$), with correct boundary values, and which are lower (or upper) directed. This means that if $u, v \in \mathcal{U}_g$ (or $\mathcal{L}_g$), then $\min(u, v) \in \mathcal{U}_g$ (or $\max(u, v) \in \mathcal{L}_g$). Then one can define upper (or lower) Perron solutions by setting $H_g(x) := \inf_{u \in \mathcal{U}_g} u(x)$ or $H_g(x) := \sup_{u \in \mathcal{L}_g} u(x)$. The natural task is of course to find suitable classes of boundary datum and related upper and lower classes providing us

- **Solvability**: Both $H_g$ and $H_g$ are solutions in $\Omega$
- **Resolutivity**: $H_g = H_g$
- **Wiener criterion**: Give a necessary and sufficient condition for the geometry of the boundary at a point $z \in \partial \Omega$ such that if $g$ is "continuous at $z$", then so are $H_g$ and $H_g$.
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Question: Can one do this in the case of the operator \((-\Delta)^{s}_p\)?
Different notions of supersolutions

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3. \((s, p)\)-viscosity supersolutions. The notion of viscosity solutions is based on the pointwise evaluation of the principal value appearing in the definition of \((-\Delta)_{p}^{s}\):

\[
(-\Delta)_{p}^{s}u(x) := \text{p.v.} \int_{\mathbb{R}^{n}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{n+sp}} \, dy.
\]
Fractional Sobolev space

We first define the natural energy spaces for the problem. The fractional Sobolev space $W^{s,p}$ is defined via Gagliardo-seminorm

$$
[u]_{W^{s,p}(\Omega)} := \left( \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \right)^{1/p}
$$

as

$$W^{s,p}(\Omega) := \{ u \in L^p(\Omega) : [u]_{W^{s,p}(\Omega)} < \infty \}$$

The local version $W^{s,p}_{\text{loc}}(\Omega)$ is defined in an obvious way.
**Tail space**

The second space controls the behavior of tails:

\[ L_{sp}^{p-1}(\mathbb{R}^n) := \left\{ u \in L_{loc}^{p-1}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \frac{|u(x)|^{p-1}}{(1 + |x|)^{n+sp}} \, dx < \infty \right\}. \]
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In particular, \( u \in L^{p-1}_{sp}(\mathbb{R}^n) \) implies that in the definition of \((-\Delta)^s u\) the nonlocal contributions are finite:

If \(|u(x)| < \infty\) and \( u \in L^{p-1}_{sp}(\mathbb{R}^n) \), then

\[
\left| \int_{\mathbb{R}^n \setminus B_r(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{n+sp}} \, dy \right| < \infty.
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Moreover, a very natural quantity

\[ \text{Tail}(u; z, r) := \left( r^{sp} \int_{\mathbb{R}^n \setminus B_r(z)} \frac{|u(x)|^{p-1}}{|x - z|^{n+sp}} \, dx \right)^{\frac{1}{p-1}} \]

appearing in the theory frequently is finite for all \( z \in \mathbb{R}^n \) and \( r > 0 \) provided that \( u \in L_{sp}^{p-1}(\mathbb{R}^n) \).
**Weak solutions: Definition**

Let us begin with the definition

**Definition**

We say that $u$ is a weak supersolution to $(-\Delta)^s_p u = 0$ in $\Omega$ if $u \in W_{loc}^{s,p}(\Omega) \cap L^{p-1}_{sp}(\mathbb{R}^n)$ satisfies

\[
\langle (-\Delta)^s_p u, \phi \rangle := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\phi(x) - \phi(y))}{|x - y|^{n+sp}} \, dx \, dy \geq 0
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whenever $\phi$ belongs to $C_0^\infty(\Omega)$ and is nonnegative. Similarly, $u$ is a weak subsolution if $-u$ is a weak supersolution, and $u$ is a weak solution if it is both sub- and supersolution.
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The definition still makes sense with measurable coefficients, i.e., if the kernel \( |x - y|^{-n-sp} \) is replaced with \( K(x, y) = a(x, y)|x - y|^{-n-sp} \), where \( a \) is symmetric, measurable and satisfies for example \( \Lambda^{-1} \leq a(x, y) \leq \Lambda \) for a.e. \( x, y \) with a constant \( \Lambda \geq 1 \).
**Initial motivation for the problem: Minimization**

Weak solutions naturally appear in minimization problems. Defining, for $g \in W^{s,p}(\mathbb{R}^n)$,

$$
\mathcal{K}_g(\Omega) := \{ u \in W^{s,p}(\mathbb{R}^n) : u = g \text{ a.e. on } \mathbb{R}^n \setminus \Omega \},
$$

then the minimizer of

$$
\min_{u \in \mathcal{K}_g(\Omega)} [u]_{W^{s,p}(\mathbb{R}^n)}
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is a weak solution in $\Omega$, with boundary values $g$, by the first variation.
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There are various ways to generalize this. For instance, letting $\Omega'$ be such that $\Omega \subset \Omega'$ and defining, for $g \in W^{s,p}(\Omega') \cap L^{p-1}_{sp}(\mathbb{R}^n)$,

$$K_g(\Omega, \Omega') := \{ u \in W^{s,p}(\Omega') \cap L^{p-1}_{sp}(\mathbb{R}^n) : u = g \text{ a.e. on } \mathbb{R}^n \setminus \Omega \},$$

it is possible to obtain existence and uniqueness of minimizers, which, in turn, are weak solutions in $\Omega$. 
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**Problem:** What can be said if $g$ does not belong to $W^{s,p}(\Omega')$ for any $\Omega' \subset \Omega$?
Weak solutions: Some properties

As the kernel is singular, there are regularization effects:

- Solutions are Hölder-continuous with measurable coefficients (Di Castro-K-Palatucci, Poincaré ’16)
- Nonlocal Harnack estimates with measurable coefficients (Di Castro-K-Palatucci, JFA ’14)
- Continuity up to the boundary in obstacle problems with measurable coefficients (Korvenpää-K-Palatucci, Calc. Var. ’16)
- If $s > p - 1$, then the gradient of solution belongs to $W^{s, p}$ for $s > 0$. (Brasco-Lindgren, AIM ’16)
- Many other recent results
- Higher regularity still open
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Weak supersolutions

Weak supersolutions play an important role in the theory.

Some basic properties of supersolutions are the following (Korvenpää-K-Palatucci, preprint):

• A weak supersolution has a lower semicontinuous (l. s. c.) representative
• Class of uniformly globally bounded weak supersolutions is closed w.r.t. pointwise convergence
• Minimum of two weak supersolutions is a weak supersolution as well
• A nonnegative supersolution \( u \) in \( B_R(z) \) satisfies

\[
\left( \int_{B_r} u^q(x) \, dx \right)^{1/q} \lesssim \text{ess inf}_{B_r} u + \left( \frac{r}{R} \right)^{sp/p-1} \text{Tail}(\max(-u, 0); z, R)
\]

whenever \( r \in (0, R/2) \). Here \( q \in \left( 0, \frac{n(p-1)}{n-ps} \right) \) for \( n > ps \) and \( q \in (0, \infty) \) for \( n \leq ps \).

Recall: \( \text{Tail}(f; z, r) := \left( r^{sp} \int_{\mathbb{R}^n \setminus B_r(z)} \frac{|f(x)|^{p-1}}{|x-z|^{n+sp}} \, dx \right)^{1/p-1} \)
(s, p)-superharmonic functions

Definition
We say that a function \( u : \mathbb{R}^n \rightarrow [−\infty, \infty] \) is an \((s, p)\)-superharmonic function in an open set \( \Omega \) if it satisfies the following four assumptions:

(i) \( u < +\infty \) almost everywhere and \( u > −\infty \) everywhere in \( \Omega \),

(ii) \( u \) is lower semicontinuous (l. s. c.) in \( \Omega \),

(iii) \( u \) satisfies the comparison in \( \Omega \) against solutions bounded from above; that is, if \( D \subset \Omega \) is an open set and \( v \in C(\overline{D}) \) is a weak solution in \( D \) such that \( \max\{v, 0\} \in L^\infty(\mathbb{R}^n) \) and \( u \geq v \) on \( \partial D \) and almost everywhere on \( \mathbb{R}^n \setminus D \), then \( u \geq v \) in \( D \),

(iv) \( u_− \) belongs to \( L^{p−1}_{sp}(\mathbb{R}^n) \).

A function \( u \) is \((s, p)\)-subharmonic in \( \Omega \) if \( −u \) is \((s, p)\)-superharmonic in \( \Omega \), and if both \( u \) and \( −u \) are \((s, p)\)-superharmonic, we say that \( u \) is \((s, p)\)-harmonic.
Properties of \((s, p)\)-superharmonic functions

Theorem (Korvenpää-K-Palatucci, preprint)

Suppose that \(u\) is \((s, p)\)-superharmonic in an open set \(\Omega\). Then it has the following properties:

(i) **Pointwise behavior.**

\[
\liminf_{y \to x} u(y) = \mathrm{ess} \liminf_{y \to x} u(y) \quad \text{for every } x \in \Omega.
\]

(ii) **Summability.** For

\[
\bar{t} := \begin{cases} 
\frac{(p-1)n}{n-sp}, & 1 < p < \frac{n}{s}, \\
+\infty, & p \geq \frac{n}{s},
\end{cases} \quad \bar{q} := \min \left\{ \frac{n(p-1)}{n-s}, p \right\},
\]

and \(h \in (0, s), \ t \in (0, \bar{t})\) and \(q \in (0, \bar{q})\),

\[u \in W_{1,0}^{h,q}(\Omega) \cap L_{\operatorname{loc}}^{t}(\Omega) \cap L_{sp}^{p-1}(\mathbb{R}^n).\]

(iii) **Connection to weak supersolutions.** If \(u\) is locally bounded in \(\Omega\) or \(u \in W_{1,0}^{s,p}(\Omega)\), then it is a weak supersolution in \(\Omega\).
Perron solutions

We may now define the upper (and lower) classes for the Perron solution

**Definition (Perron solutions)**

Let $\Omega$ be an open set. Assume that $g \in L_{sp}^{p-1}(\mathbb{R}^n)$. The upper class $\mathcal{U}_g$ of $g$ consists of all functions $u$ such that

(i) $u$ is $(s, p)$-superharmonic in $\Omega$,

(ii) $u$ is bounded from below in $\Omega$,

(iii) $\liminf_{\Omega \ni y \to x} u(y) \geq \text{ess lim sup}_{\mathbb{R}^n \setminus \Omega \ni y \to x} g(y)$ for all $x \in \partial \Omega$,

(iv) $u = g$ almost everywhere in $\mathbb{R}^n \setminus \Omega$.

The lower class is $\mathcal{L}_g := \{ u : -u \in \mathcal{U}_g \}$. Define

$$\overline{H}_g := \inf \{ u : u \in \mathcal{U}_g \} \quad \text{and} \quad \underline{H}_g := \sup \{ u : u \in \mathcal{L}_g \}.$$ 

The definition (and some work) guarantees that $\overline{H}_g \geq \underline{H}_g$. 

Theorem (Korvenpää-K-Palatucci, preprint)

The Perron solution $H_g$ can be either identically $+\infty$ in $\Omega$, identically $-\infty$ in $\Omega$, or $(s, p)$-harmonic in $\Omega$. 

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The Perron solution $\bar{H}_g$ ($\underline{H}_g$) can be either identically $+\infty$ in $\Omega$, identically $-\infty$ in $\Omega$, or $(s, p)$-harmonic in $\Omega$. 


A few examples

The function

\[ u(x) = c_{n,s} \left( 1 - |x|^2 \right)^s \int_{\mathbb{R}^n \setminus B_1(0)} g(y) (|y|^2 - 1)^{-s} |x - y|^{-n} \, dy, \quad x \in B_1(0), \]

solves \((-\Delta)^s u = 0\) in \(B_1(0)\) with \(u = g\) on \(\mathbb{R}^n \setminus B_1(0)\).
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solves $$(-\Delta)^s u = 0$$ in $$B_1(0)$$ with $$u = g$$ on $$\mathbb{R}^n \setminus B_1(0).$$

**Example**

Taking the function $$g(x) = |x|^2 - 1|^{s-1}, \; g \in L^1_{2s}(\mathbb{R}^n),$$ as boundary values in the Poisson formula above, the integral does not converge. This example suggests that in this case $$\overline{H}g \equiv \underline{H}g \equiv +\infty$$ in $$B_1(0).$$ The example also tells that one can not expect bounded solutions for all $$g \in L^1_{2s}(\mathbb{R}^n).$$
A few examples

The function

$$u(x) = c_{n,s} \left(1 - |x|^2\right)^s \int_{\mathbb{R}^n \setminus B_1(0)} g(y) \left(|y|^2 - 1\right)^{-s} |x - y|^{-n} dy, \quad x \in B_1(0),$$

solves $$(-\Delta)^s u = 0$$ in $$B_1(0)$$ with $$u = g$$ on $$\mathbb{R}^n \setminus B_1(0)$$.

Example

Taking the function $$g(x) = \left| |x|^2 - 1\right|^{s-1}, g \in L^1_{2s}(\mathbb{R}^n)$$, as boundary values in the Poisson formula above, the integral does not converge. This example suggests that in this case $$\overline{H}_g \equiv H_g \equiv +\infty$$ in $$B_1(0)$$. The example also tells that one can not expect bounded solutions for all $$g \in L^1_{2s}(\mathbb{R}^n)$$.

Example

Let us consider the previous example with $$g$$ reflected to the negative side in the half space, i. e., $$g(x) = \text{sign}(x_n)\left| |x|^2 - 1\right|^{s-1}$$. Then the “solution” via Poisson formula, for $$x \in B_1$$, is $$u(x) = \text{sign}(x_n) \cdot \infty$$, which is suggesting that we should now have $$\overline{H}_g \equiv +\infty$$ and $$\underline{H}_g \equiv -\infty$$ in $$B_1(0)$$: failure of resolutivity in $$L^1_{2s}(\mathbb{R}^n)$$?
“Viscosity” solutions

Let us finally comment another possible class for upper solutions, namely viscosity supersolutions.

**Definition**

We say that a function $u : \mathbb{R}^n \to [-\infty, \infty]$ is an $(s, p)$-viscosity supersolution in $\Omega$ if it satisfies the following four assumptions.

(i) $u < +\infty$ almost everywhere in $\mathbb{R}^n$, and $u > -\infty$ everywhere in $\Omega$.

(ii) $u$ is lower semicontinuous in $\Omega$.

(iii) If $\phi \in C^2(B_r(x_0))^3$ for some $B_r(x_0) \subseteq \Omega$ is such that $\phi(x_0) = u(x_0)$ and $\phi \leq u$ in $B_r(x_0)$, then $(-\Delta)^s_p \phi_r(x_0) \geq 0$, where

$$
\phi_r(x) = \begin{cases} 
\phi(x), & x \in B_r(x_0), \\
u(x), & x \in \mathbb{R}^n \setminus B_r(x_0).
\end{cases}
$$

(iv) $u_-$ belongs to $L^{p-1}_{sp}(\mathbb{R}^n)$.

---

3Replacing $C^2$ with Dini-$C^{sp-1}_p$ for $p > \frac{2}{2-s}$ leads to the same class. The case $1 < p \leq \frac{2}{2-s}$ needs an extra assumption on the critical set of $\phi$. 
Equivalence

It turns out that the classes of \((s, p)\)-superharmonic functions and \((s, p)\)-viscosity solutions are the same.

*Theorem (Korvenpää-K-Lindgren)*

A function \(u\) is \((s, p)\)-superharmonic in \(\Omega\) if and only if it is an \((s, p)\)-viscosity supersolution in \(\Omega\).
Equivalence

It turns out that the classes of \((s, p)\)-superharmonic functions and \((s, p)\)-viscosity solutions are the same.

**Theorem (Korvenpää-K-Lindgren)**

A function \(u\) is \((s, p)\)-superharmonic in \(\Omega\) if and only if it is an \((s, p)\)-viscosity supersolution in \(\Omega\).

As an immediate corollary we get:

**Theorem (Korvenpää-K-Lindgren)**

A function \(u\) is a continuous weak solution to \((-\Delta)^s_p u = 0\) in \(\Omega\) if and only if it is an \((s, p)\)-viscosity solution in \(\Omega\).
The proof

\[ u \text{ is } (s, p)\text{-superharmonic} \implies u \text{ is } (s, p)\text{-viscosity supersolution} \]
The proof

\( u \) is \((s, p)\)-superharmonic \(\implies\) \( u \) is \((s, p)\)-viscosity supersolution

- Let \( \phi \) be s.t. \( u \geq \phi \) in \( B_r(z) \) and \( u(z) = \phi(z) \). Assume on contrary that \((\Delta)_p^s \phi_r(z) < 0\).
The proof

\( u \) is \((s, p)\)-superharmonic \(\implies\) \( u \) is \((s, p)\)-viscosity supersolution

- Let \( \phi \) be s.t. \( u \geq \phi \) in \( B_r(z) \) and \( u(z) = \phi(z) \). Assume on contrary that \( (-\Delta)^s p \phi_r(z) < 0 \).
- \( (-\Delta)^s p (\phi_r + \delta \eta)(\cdot) < 0 \) for small enough \( \delta > 0 \) and \( \eta \) a cut-off function in the neighborhood of \( z \) by the counter assumption (this is where the most of the computations happen)
The proof

\(u\) is \((s,p)\)-superharmonic \(\iff\) \(u\) is \((s,p)\)-viscosity supersolution

- Let \(\phi\) be s.t. \(u \geq \phi\) in \(B_r(z)\) and \(u(z) = \phi(z)\). Assume on contrary that \((-\Delta)^s_p \phi_r(z) < 0\).

- \((-\Delta)^s_p (\phi_r + \delta \eta)(\cdot) < 0\) for small enough \(\delta > 0\) and \(\eta\) a cut-off function in the neighborhood of \(z\) by the counter assumption (this is where the most of the computations happen)

- Show that

\[
\left| \int_{B_r(x) \setminus B_{\varepsilon}(x)} \frac{|\phi(x) - \phi(y)|^{p-2}(\phi(x) - \phi(y))}{|x - y|^{n+sp}} \, dy \right| = O(r)
\]
The proof

$u$ is $(s, p)$-superharmonic $\implies u$ is $(s, p)$-viscosity supersolution

- Let $\phi$ be s.t. $u \geq \phi$ in $B_r(z)$ and $u(z) = \phi(z)$. Assume on contrary that $(-\Delta)^s_p \phi_r(z) < 0$.

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- “Integrate by parts” to show that $\phi_r + \delta \eta$ is a weak subsolution
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  \]

- "Integrate by parts" to show that $\phi_r + \delta \eta$ is a weak subsolution

- Use the comparison principle for weak super- and subsolutions to reach a contradiction
The proof

\[ u \text{ is } (s,p)\text{-viscosity supersolution} \implies u \text{ is } (s,p)\text{-superharmonic} \]
The proof

\[ u \text{ is } (s, p)\text{-viscosity supersolution} \implies u \text{ is } (s, p)\text{-superharmonic} \]

- Take a domain \( D \subseteq \Omega \) and a continuous weak solution \( v \) s.t. \( u \geq v \)
on \( \partial D \) and a.e. on \( \mathbb{R}^n \setminus D \).
The proof

\[ u \text{ is } (s, p)\text{-viscosity supersolution} \implies u \text{ is } (s, p)\text{-superharmonic} \]

- Take a domain \( D \subseteq \Omega \) and a continuous weak solution \( v \) s.t. \( u \geq v \) on \( \partial D \) and a.e. on \( \mathbb{R}^n \setminus D \).

- As in the first part, one can show that \( v \) is an \((s, p)\)-viscosity subsolution.
The proof

\( u \) is \((s, p)\)-viscosity supersolution \(\iff\) \( u \) is \((s, p)\)-superharmonic

- Take a domain \( D \subset \Omega \) and a continuous weak solution \( v \) s.t. \( u \geq v \) on \( \partial D \) and a.e. on \( \mathbb{R}^n \setminus D \).
- As in the first part, one can show that \( v \) is an \((s, p)\)-viscosity subsolution
- Conclude the proof by proving a comparison principle for viscosity solutions (MUCH easier than in the case of local viscosity solutions)
Thank you for your attention!